8-3 Circles

What You'll Learn

- Write equations of circles.
- Graph circles.

Vocabulary

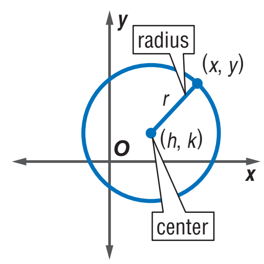
- circle
- center
- tangent

Why are circles important in air traffic control?

Radar equipment can be used to detect and locate objects that are too far away to be seen by the human eye. The radar systems at major airports can typically detect and track aircraft up to 45 to 70 miles in any direction from the airport. The boundary of the region that a radar system can monitor can be modeled by a circle.

EQUATIONS OF CIRCLES A **circle** is the set of all points in a plane that are equidistant from a given point in the plane, called the **center**. Any segment whose endpoints are the center and a point on the circle is a *radius* of the circle.

Assume that (x, y) are the coordinates of a point on the circle at the right. The center is at (h, k), and the radius is r. You can find an equation of the circle by using the Distance Formula.



$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = d \quad \text{Distance Formula}$$

$$\sqrt{(x - h)^2 + (y - k)^2} = r \quad (x_1, y_1) = (h, k),\ (x_2, y_2) = (x, y),\ d = r$$

$$(x - h)^2 + (y - k)^2 = r^2 \quad \text{Square each side.}$$

Key Concept — *Equation of a Circle*

The equation of a circle with center (h, k) and radius r units is $(x - h)^2 + (y - k)^2 = r^2$.

Example 1 Write an Equation Given the Center and Radius

NUCLEAR POWER In 1986, a nuclear reactor exploded at a power plant about 110 kilometers north and 15 kilometers west of Kiev. At first, officials evacuated people within 30 kilometers of the power plant. Write an equation to represent the boundary of the evacuated region if the origin of the coordinate system is at Kiev.

Since Kiev is at (0, 0), the power plant is at (−15, 110). The boundary of the evacuated region is the circle centered at (−15, 110) with radius 30 kilometers.

$$(x - h)^2 + (y - k)^2 = r^2 \quad \text{Equation of a circle}$$

$$[x - (-15)]^2 + (y - 110)^2 = 30^2 \quad (h, k) = (-15, 110),\ r = 30$$

$$(x + 15)^2 + (y - 110)^2 = 900 \quad \text{Simplify.}$$

The equation is $(x + 15)^2 + (y - 110)^2 = 900$.

Example 2 Write an Equation Given a Diameter

Write an equation for a circle if the endpoints of a diameter are at (5, 4) and (−2, −6).

Explore To write an equation of a circle, you must know the center and the radius.

Plan You can find the center of the circle by finding the midpoint of the diameter. Then you can find the radius of the circle by finding the distance from the center to one of the given points.

Solve First find the center of the circle.

$$(h, k) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) \quad \text{Midpoint Formula}$$

$$= \left(\frac{5 + (-2)}{2}, \frac{4 + (-6)}{2}\right) \quad (x_1, y_1) = (5, 4), (x_2, y_2) = (-2, -6)$$

$$= \left(\frac{3}{2}, \frac{-2}{2}\right) \quad \text{Add.}$$

$$= \left(\frac{3}{2}, -1\right) \quad \text{Simplify.}$$

Now find the radius.

$$r = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \quad \text{Distance Formula}$$

$$= \sqrt{\left(\frac{3}{2} - 5\right)^2 + (-1 - 4)^2} \quad (x_1, y_1) = (5, 4), (x_2, y_2) = \left(\frac{3}{2}, -1\right)$$

$$= \sqrt{\left(-\frac{7}{2}\right)^2 + (-5)^2} \quad \text{Subtract.}$$

$$= \sqrt{\frac{149}{4}} \quad \text{Simplify.}$$

The radius of the circle is $\sqrt{\frac{149}{4}}$ units, so $r^2 = \frac{149}{4}$.

Substitute h, k, and r^2 into the standard form of the equation of a circle.

An equation of the circle is $\left(x - \frac{3}{2}\right)^2 + [y - (-1)]^2 = \frac{149}{4}$ or $\left(x - \frac{3}{2}\right)^2 + (y + 1)^2 = \frac{149}{4}$.

Examine Each of the given points satisfies the equation, so the equation is reasonable.

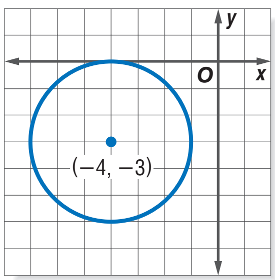
A line in the plane of a circle can intersect the circle in zero, one, or two points. A line that intersects the circle in exactly one point is said to be **tangent** to the circle. The line and the circle are tangent to each other at this point.

Example 3 Write an Equation Given the Center and a Tangent

Write an equation for a circle with center at (−4, −3) that is tangent to the *x*-axis.

Sketch the circle. Since the circle is tangent to the *x*-axis, its radius is 3.

An equation of the circle is $(x + 4)^2 + (y + 3)^2 = 9$.



GRAPH CIRCLES You can use completing the square, symmetry, and transformations to help you graph circles. The equation $(x - h)^2 + (y - k)^2 = r^2$ is obtained from the equation $x^2 + y^2 = r^2$ by replacing x with $x - h$ and y with $y - k$. So, the graph of $(x - h)^2 + (y - k)^2 = r^2$ is the graph of $x^2 + y^2 = r^2$ translated h units to the right and k units up.

Example 4 Graph an Equation in Standard Form

Find the center and radius of the circle with equation $x^2 + y^2 = 25$. Then graph the circle.

The center of the circle is at (0, 0), and the radius is 5.

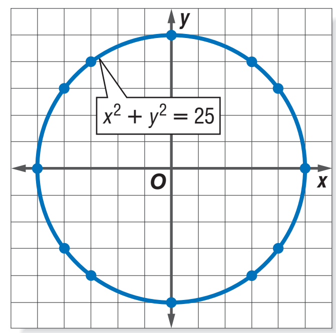
x	y
0	5
3	4
4	3
5	0

The table lists some integer values for x and y that satisfy the equation.

Since the circle is centered at the origin, it is symmetric about the y-axis. Therefore, the points at $(-3, 4)$, $(-4, 3)$ and $(-5, 0)$ lie on the graph.

The circle is also symmetric about the x-axis, so the points at $(-4, -3)$, $(-3, -4)$, $(0, -5)$, $(3, -4)$, and $(4, -3)$ lie on the graph.

Graph all of these points and draw the circle that passes through them.



Example 5 Graph an Equation Not in Standard Form

Find the center and radius of the circle with equation $x^2 + y^2 - 4x + 8y - 5 = 0$. Then graph the circle.

Complete the squares.

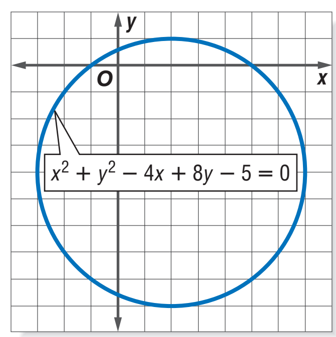
$$x^2 + y^2 - 4x + 8y - 5 = 0$$

$$x^2 - 4x + \blacksquare + y^2 + 8y + \blacksquare = 5 + \blacksquare + \blacksquare$$

$$x^2 - 4x + 4 + y^2 + 8y + 16 = 5 + 4 + 16$$

$$(x - 2)^2 + (y + 4)^2 = 25$$

The center of the circle is at $(2, -4)$, and the radius is 5. In the equation from Example 4, x has been replaced by $x - 2$, and y has been replaced by $y + 4$. The graph is the graph from Example 4 translated 2 units to the right and down 4 units.



Check for Understanding

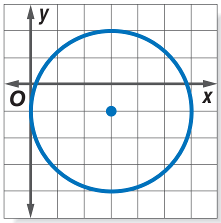
Concept Check

1. **OPEN ENDED** Write an equation for a circle with center at $(6, -2)$.
2. Write $x^2 + y^2 + 6x - 2y - 54 = 0$ in standard form by completing the square. Describe the transformation that can be applied to the graph of $x^2 + y^2 = 64$ to obtain the graph of the given equation.
3. **FIND THE ERROR** Juwan says that the circle with equation $(x - 4)^2 + y^2 = 36$ has radius 36 units. Lucy says that the radius is 6 units. Who is correct? Explain your reasoning.

CONTENTS

Guided Practice

4. Write an equation for the graph at the right.



Write an equation for the circle that satisfies each set of conditions.

5. center $(-1, -5)$, radius 2 units
6. endpoints of a diameter at $(-4, 1)$ and $(4, -5)$
7. center $(3, -7)$, tangent to the y-axis

Find the center and radius of the circle with the given equation. Then graph the circle.

8. $(x - 4)^2 + (y - 1)^2 = 9$
9. $x^2 + (y - 14)^2 = 34$
10. $(x - 4)^2 + y^2 = \frac{16}{25}$
11. $\left(x + \frac{2}{3}\right)^2 + \left(y - \frac{1}{2}\right)^2 = \frac{8}{9}$
12. $x^2 + y^2 + 8x - 6y = 0$
13. $x^2 + y^2 + 4x - 8 = 0$

Application

AEROSPACE **For Exercises 14 and 15, use the following information.**
In order for a satellite to remain in a circular orbit above the same spot on Earth, the satellite must be 35,800 kilometers above the equator.

14. Write an equation for the orbit of the satellite. Use the center of Earth as the origin and 6400 kilometers for the radius of Earth.
15. Draw a labeled sketch of Earth and the orbit to scale.

Practice and Apply

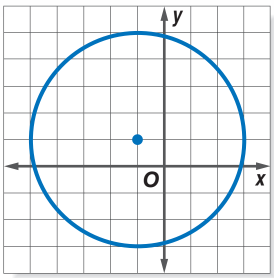
Homework Help

For Exercises	See Examples
16–29	1–3
30–48	4, 5

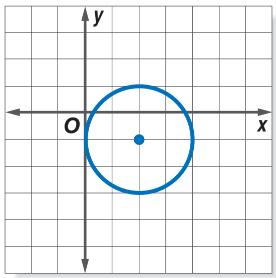
Extra Practice
See page 845.

Write an equation for each graph.

16.



17.



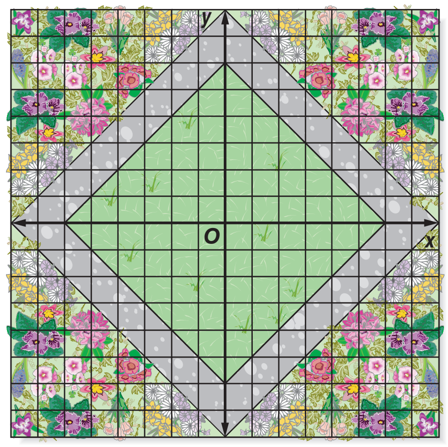
Write an equation for the circle that satisfies each set of conditions.

18. center $(0, 3)$, radius 7 units
19. center $(-8, 7)$, radius $\frac{1}{2}$ unit
20. endpoints of a diameter at $(-5, 2)$ and $(3, 6)$
21. endpoints of a diameter at $(11, 18)$ and $(-13, -19)$
22. center $(8, -9)$, passes through $(21, 22)$
23. center $\left(-\sqrt{13}, 42\right)$, passes through the origin
24. center at $(-8, -7)$, tangent to y-axis
25. center at $(4, 2)$, tangent to x-axis
26. center in the first quadrant; tangent to $x = -3$, $x = 5$, and the x-axis
27. center in the second quadrant; tangent to $y = -1$, $y = 9$, and the y-axis

WebQuest

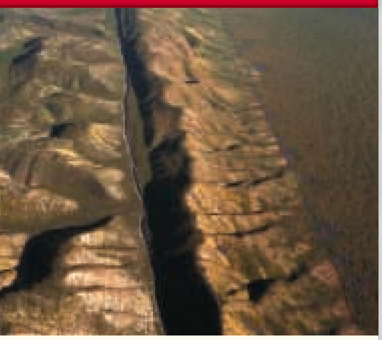
The epicenter of an earthquake can be located by using the equation of a circle. Visit www.algebra2.com/webquest to continue work on your WebQuest project.

28. **LANDSCAPING** The design of a garden is shown at the right. A pond is to be built in the center region. What is the equation of the largest circular pond centered at the origin that would fit within the walkways?



29. **EARTHQUAKES** The University of Southern California is located about 2.5 miles west and about 2.8 miles south of downtown Los Angeles. Suppose an earthquake occurs with its epicenter about 40 miles from the university. Assume that the origin of a coordinate plane is located at the center of downtown Los Angeles. Write an equation for the set of points that could be the epicenter of the earthquake.

More About. . .



Earthquakes

Southern California has about 10,000 earthquakes per year many of which occur at or near the San Andreas fault. Most are too small to be felt.

Source: www.earthquake.usgs.gov

Find the center and radius of the circle with the given equation. Then graph the circle.

30. $x^2 + (y + 2)^2 = 4$

31. $x^2 + y^2 = 144$

32. $(x - 3)^2 + (y - 1)^2 = 25$

33. $(x + 3)^2 + (y + 7)^2 = 81$

34. $(x - 3)^2 + y^2 = 16$

35. $(x - 3)^2 + (y + 7)^2 = 50$

36. $\left(x + \sqrt{5}\right)^2 + y^2 - 8y = 9$

37. $x^2 + \left(y - \sqrt{3}\right)^2 + 4x = 25$

38. $x^2 + y^2 + 6y = -50 - 14x$

39. $x^2 + y^2 - 6y - 16 = 0$

40. $x^2 + y^2 + 2x - 10 = 0$

41. $x^2 + y^2 - 18x - 18y + 53 = 0$

42. $x^2 + y^2 + 9x - 8y + 4 = 0$

43. $x^2 + y^2 - 3x + 8y = 20$

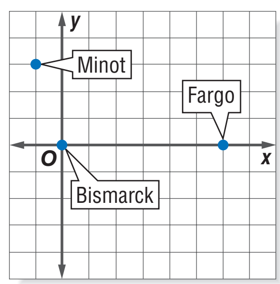
44. $x^2 - 12x + 84 = -y^2 + 16y$

45. $x^2 + y^2 + 2x + 4y = 9$

46. $3x^2 + 3y^2 + 12x - 6y + 9 = 0$

47. $4x^2 + 4y^2 + 36y + 5 = 0$

48. **RADIO** The diagram at the right shows the relative locations of some cities in North Dakota. The x-axis represents Interstate 94. The scale is 1 unit = 30 miles. While driving west on the highway, Doralina is listening to a radio station in Minot. She estimates the range of the signal to be 120 miles. How far west of Bismarck will she be able to pick up the signal?



49. **CRITICAL THINKING** A circle has its center on the line with equation $y = 2x$. The circle passes through $(1, -3)$ and has a radius of $\sqrt{5}$ units. Write an equation of the circle.

50. **WRITING IN MATH** Answer the question that was posed at the beginning of the lesson.

Why are circles important in air traffic control?

Include the following in your answer:

- an equation of the circle that determines the boundary of the region where planes can be detected if the range of the radar is 50 miles and the radar is at the origin, and
- how an air traffic controller's job would be different for a region whose boundary is modeled by $x^2 + y^2 = 4900$ instead of $x^2 + y^2 = 1600$.

Standardized Test Practice

51. Find the radius of the circle with equation $x^2 + y^2 + 8x + 8y + 28 = 0$.

(A) 2 (B) 4 (C) 8 (D) 28

52. Find the center of the circle with equation $x^2 + y^2 - 10x + 6y + 27 = 0$.

(A) $(-10, 6)$ (B) $(1, 1)$ (C) $(10, -6)$ (D) $(5, -3)$

Graphing Calculator

CIRCLES **For Exercises 53–56, use the following information.**
Since a circle is not the graph of a function, you cannot enter its equation directly into a graphing calculator. Instead, you must solve the equation for y. The result will contain a $\pm$ symbol, so you will have two functions.

53. Solve $(x + 3)^2 + y^2 = 16$ for y.

54. What two functions should you enter to graph the given equation?

55. Graph $(x + 3)^2 + y^2 = 16$ on a graphing calculator.

56. Solve $(x + 3)^2 + y^2 = 16$ for x. What parts of the circle do the two expressions for x represent?

Maintain Your Skills

Mixed Review

Identify the coordinates of the vertex and focus, the equations of the axis of symmetry and directrix, and the direction of opening of the parabola with the given equation. Then find the length of the latus rectum and graph the parabola. *(Lesson 8-2)*

57. $x = -3y^2 + 1$ **58.** $y + 2 = -(x - 3)^2$ **59.** $y = x^2 + 4x$

Find the midpoint of the line segment with endpoints at the given coordinates. *(Lesson 8-1)*

60. $(5, -7), (3, -1)$ **61.** $(2, -9), (-4, 5)$ **62.** $(8, 0), (-5, 12)$

Find all of the rational zeros for each function. *(Lesson 7-5)*

63. $f(x) = x^3 + 5x^2 + 2x - 8$ **64.** $g(x) = 2x^3 - 9x^2 + 7x + 6$

65. PHOTOGRAPHY The perimeter of a rectangular picture is 86 inches. Twice the width exceeds the length by 2 inches. What are the dimensions of the picture? *(Lesson 3-2)*

Getting Ready for the Next Lesson

PREREQUISITE SKILL **Solve each equation. Assume that all variables are positive.**
(To review ***solving quadratic equations****, see Lesson 6-4.)*

66. $c^2 = 13^2 - 5^2$ **67.** $c^2 = 10^2 - 8^2$ **68.** $\left(\sqrt{7}\right)^2 = a^2 - 3^2$

69. $24^2 = a^2 - 7^2$ **70.** $4^2 = 6^2 - b^2$ **71.** $\left(2\sqrt{14}\right)^2 = 8^2 - b^2$

Practice Quiz 1

Lessons 8-1 through 8-3

Find the distance between each pair of points with the given coordinates. *(Lesson 8-1)*

1. $(9, 5), (4, -7)$ **2.** $(0, -5), (10, -3)$

Identify the coordinates of the vertex and focus, the equations of the axis of symmetry and directrix, and the direction of opening of the parabola with the given equation. Then find the length of the latus rectum and graph the parabola. *(Lesson 8-2)*

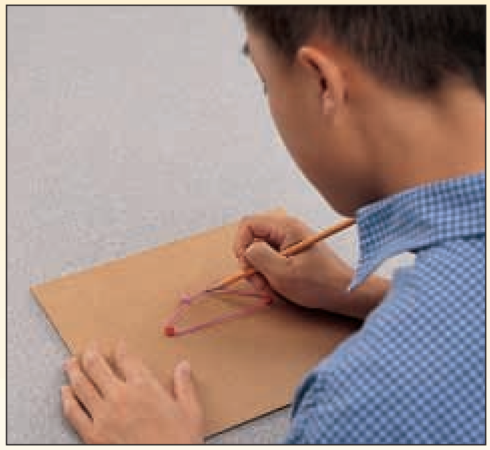
3. $y^2 = 6x$ **4.** $y = x^2 + 8x + 20$

5. Find the center and radius of the circle with equation $x^2 + (y - 4)^2 = 49$. Then graph the circle. *(Lesson 8-3)*

Algebra Activity

A Preview of Lesson 8-4

Investigating Ellipses



Follow the steps below to construct another type of conic section.

Step 1 Place two thumbtacks in a piece of cardboard, about 1 foot apart.

Step 2 Tie a knot in a piece of string and loop it around the thumbtacks.

Step 3 Place your pencil in the string. Keep the string tight and draw a curve.

Step 4 Continue drawing until you return to your starting point.

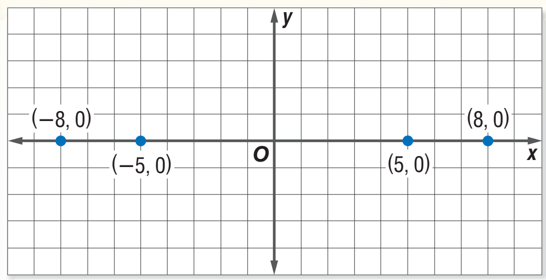
The curve you have drawn is called an **ellipse**. The points where the thumbtacks are located are called the **foci** of the ellipse. *Foci* is the plural of *focus*.

Model and Analyze

Place a large piece of grid paper on a piece of cardboard.

1. Place the thumbtacks at (8, 0) and (−8, 0). Choose a string long enough to loop around both thumbtacks. Draw an ellipse.

2. Repeat Exercise 1, but place the thumbtacks at (5, 0) and (−5, 0). Use the same loop of string and draw an ellipse. How does this ellipse compare to the one in Exercise 1?



Place the thumbtacks at each set of points and draw an ellipse.

You may change the length of the loop of string if you like.

3. (12, 0), (−12, 0) **4.** (2, 0), (−2, 0) **5.** (14, 4), (−10, 4)

Make a Conjecture

In Exercises 6–10, describe what happens to the shape of an ellipse when each change is made.

6. The thumbtacks are moved closer together.

7. The thumbtacks are moved farther apart.

8. The length of the loop of string is increased.

9. The thumbtacks are arranged vertically.

10. One thumbtack is removed, and the string is looped around the remaining thumbtack.

11. Pick a point on one of the ellipses you have drawn. Use a ruler to measure the distances from that point to the points where the thumbtacks were located. Add the distances. Repeat for other points on the same ellipse. What relationship do you notice?

12. Could this activity be done with a rubber band instead of a piece of string? Explain.

8-4 Ellipses

What You'll Learn

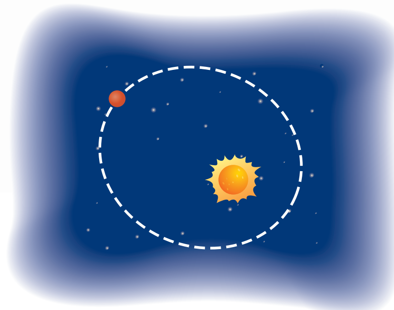
- Write equations of ellipses.
- Graph ellipses.

Vocabulary

- ellipse
- foci
- major axis
- minor axis
- center

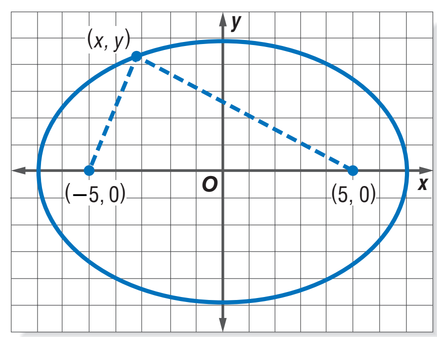
Why are ellipses important in the study of the solar system?

Fascination with the sky has caused people to wonder, observe, and make conjectures about the planets since the beginning of history. Since the early 1600s, the orbits of the planets have been known to be ellipses with the Sun at a focus.



EQUATIONS OF ELLIPSES As you discovered in the Algebra Activity on page 432, an **ellipse** is the set of all points in a plane such that the sum of the distances from two fixed points is constant. The two fixed points are called the **foci** of the ellipse.

The ellipse at the right has foci at (5, 0) and (−5, 0). The distances from either of the x-intercepts to the foci are 2 units and 12 units, so the sum of the distances from any point with coordinates (x, y) on the ellipse to the foci is 14 units.



You can use the Distance Formula and the definition of an ellipse to find an equation of this ellipse.

$$\underbrace{\text{The distance between } (x, y) \text{ and } (-5, 0)} + \underbrace{\text{the distance between } (x, y) \text{ and } (5, 0)} = 14.$$

$$\sqrt{(x+5)^2 + y^2} + \sqrt{(x-5)^2 + y^2} = 14$$

$$\sqrt{(x+5)^2 + y^2} = 14 - \sqrt{(x-5)^2 + y^2} \quad \text{Isolate the radicals.}$$

$$(x+5)^2 + y^2 = 196 - 28\sqrt{(x-5)^2 + y^2} + (x-5)^2 + y^2 \quad \text{Square each side.}$$

$$x^2 + 10x + 25 + y^2 = 196 - 28\sqrt{(x-5)^2 + y^2} + x^2 - 10x + 25 + y^2$$

$$20x - 196 = -28\sqrt{(x-5)^2 + y^2} \quad \text{Simplify.}$$

$$5x - 49 = -7\sqrt{(x-5)^2 + y^2} \quad \text{Divide each side by 4.}$$

$$25x^2 - 490x + 2401 = 49[(x-5)^2 + y^2] \quad \text{Square each side.}$$

$$25x^2 - 490x + 2401 = 49x^2 - 490x + 1225 + 49y^2 \quad \text{Distributive Property}$$

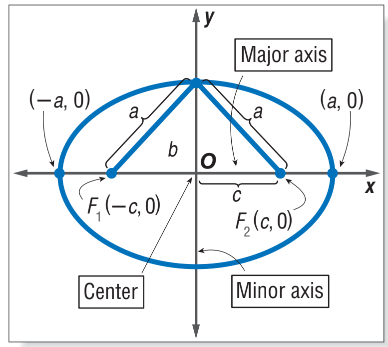
$$-24x^2 - 49y^2 = -1176 \quad \text{Simplify.}$$

$$\frac{x^2}{49} + \frac{y^2}{24} = 1 \quad \text{Divide each side by } -1176.$$

An equation for this ellipse is $\frac{x^2}{49} + \frac{y^2}{24} = 1$.

Every ellipse has two axes of symmetry. The points at which the ellipse intersects its axes of symmetry determine two segments with endpoints on the ellipse called the **major axis** and the **minor axis**. The axes intersect at the **center** of the ellipse. The foci of an ellipse always lie on the major axis.

Study the ellipse at the right. The sum of the distances from the foci to any point on the ellipse is the same as the length of the major axis, or $2a$ units. The distance from the center to either focus is c units. By the Pythagorean Theorem, a, b, and c are related by the equation $c^2 = a^2 - b^2$. Notice that the x- and y-intercepts, $(\pm a, 0)$ and $(0, \pm b)$, satisfy the quadratic equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. This is the standard form of the equation of an ellipse with its center at the origin and a horizontal major axis.



Study Tip

Vertices of Ellipses
The endpoints of each axis are called the *vertices* of the ellipse.

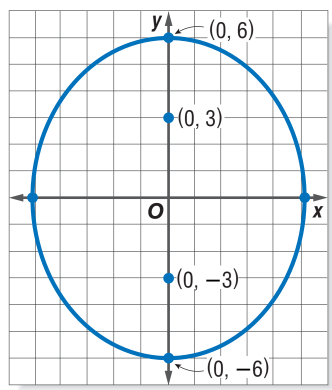
Key Concept — Equations of Ellipses with Centers at the Origin

Standard Form of Equation	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	$\frac{y^2}{a^2} + \frac{x^2}{b^2} = 1$
Direction of Major Axis	horizontal	vertical
Foci	$(c, 0), (-c, 0)$	$(0, c), (0, -c)$
Length of Major Axis	$2a$ units	$2a$ units
Length of Minor Axis	$2b$ units	$2b$ units

In either case, $a^2 \geq b^2$ and $c^2 = a^2 - b^2$. You can determine if the foci are on the x-axis or the y-axis by looking at the equation. If the x^2 term has the greater denominator, the foci are on the x-axis. If the y^2 term has the greater denominator, the foci are on the y-axis.

Example 1 Write an Equation for a Graph

Write an equation for the ellipse shown at the right.



In order to write the equation for the ellipse, we need to find the values of a and b for the ellipse. We know that the length of the major axis of any ellipse is $2a$ units. In this ellipse, the length of the major axis is the distance between the points at $(0, 6)$ and $(0, -6)$. This distance is 12 units.

$2a = 12$ Length of major axis $= 12$

$a = 6$ Divide each side by 2.

The foci are located at $(0, 3)$ and $(0, -3)$, so $c = 3$. We can use the relationship between a, b, and c to determine the value of b.

$c^2 = a^2 - b^2$ Equation relating a, b, and c

$9 = 36 - b^2$ $c = 3$ and $a = 6$

$b^2 = 27$ Solve for b^2.

Since the major axis is vertical, substitute 36 for a^2 and 27 for b^2 in the form $\frac{y^2}{a^2} + \frac{x^2}{b^2} = 1$. An equation of the ellipse is $\frac{y^2}{36} + \frac{x^2}{27} = 1$.

Example 2 Write an Equation Given the Lengths of the Axes

MUSEUMS **In an ellipse, sound or light coming from one focus is reflected to the other focus. In a whispering gallery, a person can hear another person whisper from across the room if the two people are standing at the foci. The whispering gallery at the Museum of Science and Industry in Chicago has an elliptical cross section that is 13 feet 6 inches by 47 feet 4 inches.**

a. Write an equation to model this ellipse. Assume that the center is at the origin and the major axis is horizontal.

The length of the major axis is $47\frac{1}{3}$ or $\frac{142}{3}$ feet.

$2a = \frac{142}{3}$ Length of major axis $= \frac{142}{3}$

$a = \frac{71}{3}$ Divide each side by 2.

The length of the minor axis is $13\frac{1}{2}$ or $\frac{27}{2}$ feet.

$2b = \frac{27}{2}$ Length of minor axis $= \frac{27}{2}$

$b = \frac{27}{4}$ Divide each side by 2.

Substitute $a = \frac{71}{3}$ and $b = \frac{27}{4}$ into the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. An equation of the ellipse is $\frac{x^2}{\left(\frac{71}{3}\right)^2} + \frac{y^2}{\left(\frac{27}{4}\right)^2} = 1$.

b. How far apart are the points at which two people should stand to hear each other whisper?

People should stand at the two foci of the ellipse. The distance between the foci is $2c$ units.

$c^2 = a^2 - b^2$ Equation relating a, b, and c

$c = \sqrt{a^2 - b^2}$ Take the square root of each side.

$2c = 2\sqrt{a^2 - b^2}$ Multiply each side by 2.

$2c = 2\sqrt{\left(\frac{71}{3}\right)^2 - \left(\frac{27}{4}\right)^2}$ Substitute $a = \frac{71}{3}$ and $b = \frac{27}{4}$.

$2c \approx 45.37$ Use a calculator.

The points where two people should stand to hear each other whisper are about 45.37 feet or 45 feet 4 inches apart.

More About. . .

Museums

The whispering gallery at Chicago's Museum of Science and Industry has a parabolic dish at each focus to help collect sound.

Source: www.msichicago.org

GRAPH ELLIPSES As with circles, you can use completing the square, symmetry, and transformations to help graph ellipses. An ellipse with its center at the origin is represented by an equation of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ or $\frac{y^2}{a^2} + \frac{x^2}{b^2} = 1$. The ellipse could be translated h units to the right and k units up. This would move the center to the point (h, k). Such a move would be equivalent to replacing x with $x - h$ and replacing y with $y - k$.

Key Concept — *Equations of Ellipses with Centers at (h, k)*

Standard Form of Equation	$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$	$\frac{(y-k)^2}{a^2} + \frac{(x-h)^2}{b^2} = 1$
Direction of Major Axis	horizontal	vertical
Foci	$(h \pm c, k)$	$(h, k \pm c)$

Example 3 Graph an Equation in Standard Form

Find the coordinates of the center and foci and the lengths of the major and minor axes of the ellipse with equation $\frac{x^2}{16} + \frac{y^2}{4} = 1$. Then graph the ellipse.

The center of this ellipse is at (0, 0).

Since $a^2 = 16$, $a = 4$. Since $b^2 = 4$, $b = 2$.

The length of the major axis is 2(4) or 8 units, and the length of the minor axis is 2(2) or 4 units. Since the x^2 term has the greater denominator, the major axis is horizontal.

$c^2 = a^2 - b^2$ — Equation relating a, b, and c

$c^2 = 4^2 - 2^2$ or 12 — $a = 4$, $b = 2$

$c = \sqrt{12}$ or $2\sqrt{3}$ — Take the square root of each side.

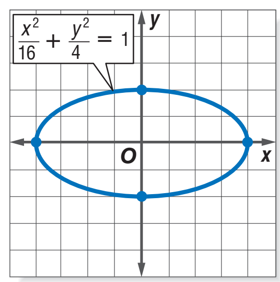
The foci are at $(2\sqrt{3}, 0)$ and $(-2\sqrt{3}, 0)$.

You can use a calculator to find some approximate nonnegative values for x and y that satisfy the equation. Since the ellipse is centered at the origin, it is symmetric about the y-axis. Therefore, the points at (−4, 0), (−3, 1.3), (−2, 1.7), and (−1, 1.9) lie on the graph.

x	y
0	2.0
1	1.9
2	1.7
3	1.3
4	0.0

The ellipse is also symmetric about the x-axis, so the points at (−3, −1.3), (−2, −1.7), (−1, −1.9), (0, −2), (1, −1.9), (2, −1.7), and (3, −1.3) lie on the graph.

Graph the intercepts, (−4, 0), (4, 0), (0, 2), and (0, −2), and draw the ellipse that passes through them and the other points.



Study Tip

Graphing Calculator

You can graph an ellipse on a graphing calculator by first solving for y. Then graph the two equations that result on the same screen.

If you are given an equation of an ellipse that is not in standard form, write it in standard form first. This will make graphing the ellipse easier.

Example 4 Graph an Equation Not in Standard Form

Find the coordinates of the center and foci and the lengths of the major and minor axes of the ellipse with equation $x^2 + 4y^2 + 4x - 24y + 24 = 0$. Then graph the ellipse.

Complete the square for each variable to write this equation in standard form.

$x^2 + 4y^2 + 4x - 24y + 24 = 0$ — Original equation

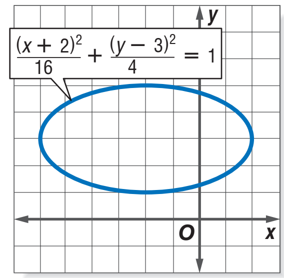
$(x^2 + 4x + \blacksquare) + 4(y^2 - 6y + \blacksquare) = -24 + \blacksquare + 4(\blacksquare)$ — Complete the squares.

$(x^2 + 4x + 4) + 4(y^2 - 6y + 9) = -24 + 4 + 4(9)$ — $\left(\frac{4}{2}\right)^2 = 4$, $\left(\frac{-6}{2}\right)^2 = 9$

$(x + 2)^2 + 4(y - 3)^2 = 16$ — Write the trinomials as perfect squares.

$\frac{(x + 2)^2}{16} + \frac{(y - 3)^2}{4} = 1$ — Divide each side by 16.

The graph of this ellipse is the graph from Example 3 translated 2 units to the left and up 3 units. The center is at $(-2, 3)$ and the foci are at $\left(-2 + 2\sqrt{3}, 0\right)$ and $\left(-2 - 2\sqrt{3}, 0\right)$. The length of the major axis is still 8 units, and the length of the minor axis is still 4 units.



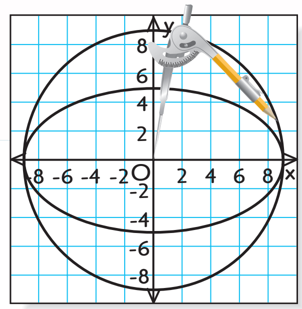
You can use a circle to locate the foci on the graph of a given ellipse.

Algebra Activity

Locating Foci

You can locate the foci of an ellipse by using the following method.

Step 1 Graph an ellipse so that its center is at the origin. Let the endpoints of the major axis be at $(-9, 0)$ and $(9, 0)$, and let the endpoints of the minor axis be at $(0, -5)$ and $(0, 5)$.



Step 2 Use a compass to draw a circle with center at $(0, 0)$ and radius 9 units.

Step 3 Draw the line with equation $y = 5$ and mark the points at which the line intersects the circle.

Step 4 Draw perpendicular lines from the points of intersection to the x-axis. The foci of the ellipse are located at the points where the perpendicular lines intersect the x-axis.

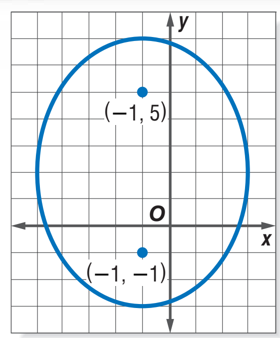
Make a Conjecture

Draw another ellipse and locate its foci. Why does this method work?

Check for Understanding

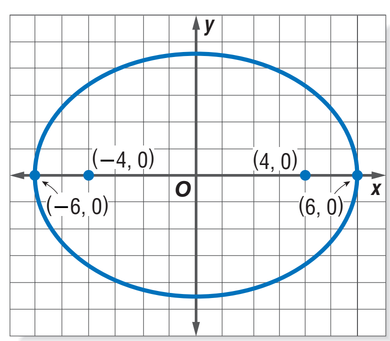
Concept Check

1. **Identify** the axes of symmetry of the ellipse at the right.
2. **Explain** why a circle is a special case of an ellipse.
3. **OPEN ENDED** Write an equation for an ellipse with its center at $(2, -5)$ and a horizontal major axis.



Guided Practice

4. Write an equation for the ellipse shown at the right.



Write an equation for the ellipse that satisfies each set of conditions.

5. endpoints of major axis at $(2, 2)$ and $(2, -10)$, endpoints of minor axis at $(0, -4)$ and $(4, -4)$
6. endpoints of major axis at $(0, 10)$ and $(0, -10)$, foci at $(0, 8)$ and $(0, -8)$

Find the coordinates of the center and foci and the lengths of the major and minor axes for the ellipse with the given equation. Then graph the ellipse.

7. $\frac{y^2}{18} + \frac{x^2}{9} = 1$

8. $\frac{(x-1)^2}{20} + \frac{(y+2)^2}{4} = 1$

9. $4x^2 + 8y^2 = 32$

10. $x^2 + 25y^2 - 8x + 100y + 91 = 0$

Application

11. **ASTRONOMY** At its closest point, Mercury is 29.0 million miles from the center of the Sun. At its farthest point, Mercury is 43.8 million miles from the center of the Sun. Write an equation for the orbit of Mercury, assuming that the center of the orbit is the origin and the Sun lies on the x-axis.

Practice and Apply

Homework Help

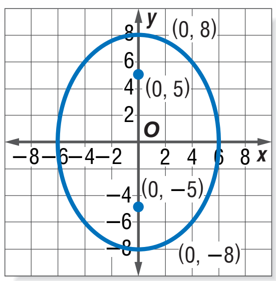
For Exercises	See Examples
12–24	1, 2
25–38	3, 4

Extra Practice

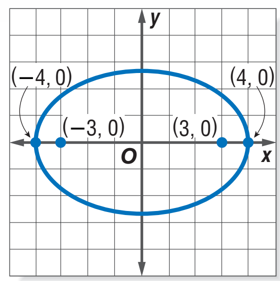
See page 846.

Write an equation for each ellipse.

12.



13.



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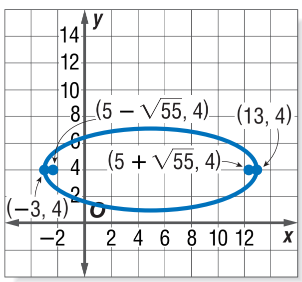
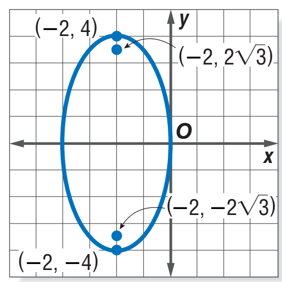


15.



Write an equation for the ellipse that satisfies each set of conditions.

16. endpoints of major axis at $(-11, 5)$ and $(7, 5)$, endpoints of minor axis at $(-2, 9)$ and $(-2, 1)$

17. endpoints of major axis at $(2, 12)$ and $(2, -4)$, endpoints of minor axis at $(4, 4)$ and $(0, 4)$

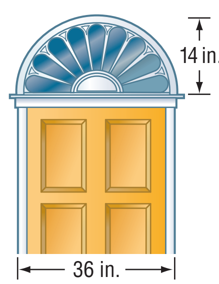
18. major axis 20 units long and parallel to y-axis, minor axis 6 units long, center at $(4, 2)$

19. major axis 16 units long and parallel to x-axis, minor axis 9 units long, center at $(5, 4)$

20. endpoints of major axis at $(10, 2)$ and $(-8, 2)$, foci at $(6, 2)$ and $(-4, 2)$

21. endpoints of minor axis at $(0, 5)$ and $(0, -5)$, foci at $(12, 0)$ and $(-12, 0)$

22. **INTERIOR DESIGN** The rounded top of the window is the top half of an ellipse. Write an equation for the ellipse if the origin is at the midpoint of the bottom edge of the window.



CONTENTS

23. **ASTRONOMY** At its closest point, Mars is 128.5 million miles from the Sun. At its farthest point, Mars is 155.0 million miles from the Sun. Write an equation for the orbit of Mars. Assume that the center of the orbit is the origin, the Sun lies on the x-axis, and the radius of the Sun is 400,000 miles.

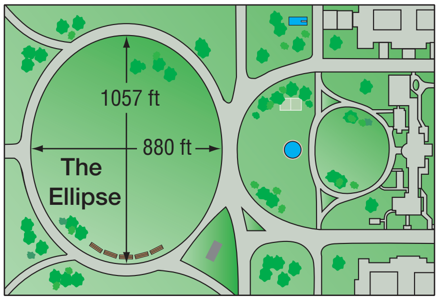
More About. . .

White House

The Ellipse, also known as President's Park South, has an area of about 16 acres.

Source: www.nps.gov

24. **WHITE HOUSE** There is an open area south of the White House known as the Ellipse. Write an equation to model the Ellipse. Assume that the origin is at the center of the Ellipse.



25. Write the equation $10x^2 + 2y^2 = 40$ in standard form.

26. What is the standard form of the equation $x^2 + 6y^2 - 2x + 12y - 23 = 0$?

Find the coordinates of the center and foci and the lengths of the major and minor axes for the ellipse with the given equation. Then graph the ellipse.

27. $\frac{y^2}{10} + \frac{x^2}{5} = 1$

28. $\frac{x^2}{25} + \frac{y^2}{9} = 1$

29. $\frac{(x+8)^2}{144} + \frac{(y-2)^2}{81} = 1$

30. $\frac{(y+11)^2}{144} + \frac{(x-5)^2}{121} = 1$

31. $3x^2 + 9y^2 = 27$

32. $27x^2 + 9y^2 = 81$

33. $16x^2 + 9y^2 = 144$

34. $36x^2 + 81y^2 = 2916$

35. $3x^2 + y^2 + 18x - 2y + 4 = 0$

36. $x^2 + 5y^2 + 4x - 70y + 209 = 0$

37. $7x^2 + 3y^2 - 28x - 12y = -19$

38. $16x^2 + 25y^2 + 32x - 150y = 159$

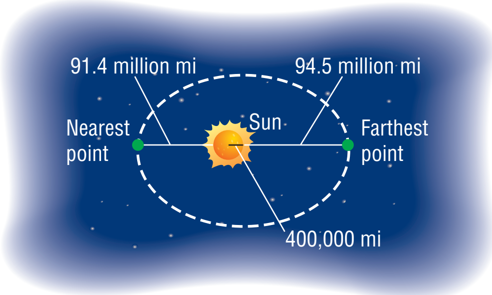
39. **CRITICAL THINKING** Find an equation for the ellipse with foci at $(\sqrt{3}, 0)$ and $(-\sqrt{3}, 0)$ that passes through $(0, 3)$.

40. **WRITING IN MATH** Answer the question that was posed at the beginning of the lesson.

Why are ellipses important in the study of the solar system?

Include the following in your answer:

- why an equation that is an accurate model of the path of a planet might be useful, and
- the distance from the center of Earth's orbit to the center of the Sun given that the Sun is at a focus of the orbit of Earth. Use the information in the figure at the right.

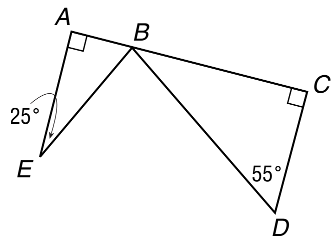


Standardized Test Practice

41. In the figure, A, B, and C are collinear. What is the measure of $\angle DBE$?

(A) 40° (B) 65°

(C) 80° (D) 100°



www.algebra2.com/self_check_quiz

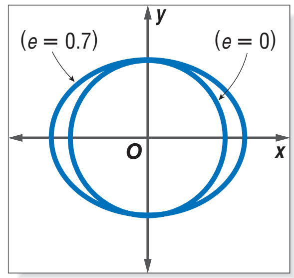
CONTENTS

42. $\sqrt{25 + 144} =$

(A) 7　　(B) 13　　(C) 17　　(D) 169

Extending the Lesson

43. **ASTRONOMY** In an ellipse, the ratio $\frac{c}{a}$ is called the **eccentricity** and is denoted by the letter e. Eccentricity measures the elongation of an ellipse. As shown in the graph at the right, the closer e is to 0, the more an ellipse looks like a circle. Pluto has the most eccentric orbit in our solar system with $e \approx 0.25$. Find an equation to model the orbit of Pluto, given that the length of the major axis is about 7.34 billion miles. Assume that the major axis is horizontal and that the center of the orbit is the origin.



Maintain Your Skills

Mixed Review

Write an equation for the circle that satisfies each set of conditions. *(Lesson 8-3)*

44. center $(3, -2)$, radius 5 units

45. endpoints of a diameter at $(5, -9)$ and $(3, 11)$

46. center $(-1, 0)$, passes through $(2, -6)$

47. center $(4, -1)$, tangent to y-axis

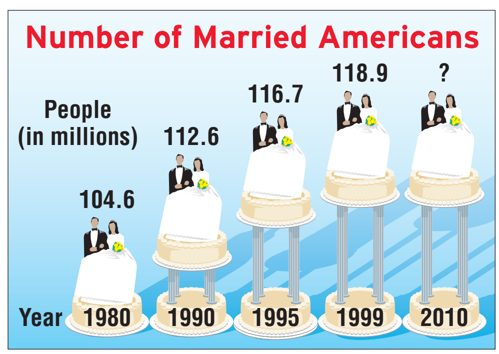
48. Write an equation of a parabola with vertex $(3, 1)$ and focus $\left(3, 1\frac{1}{2}\right)$. Then draw the graph. *(Lesson 8-2)*

MARRIAGE **For Exercises 49–51, use the table at the right that shows the number of married Americans over the last few decades.** *(Lesson 2-5)*

49. Draw a scatter plot in which x is the number of years since 1980.

50. Find a prediction equation.

51. Predict the number of married Americans in 2010.



Source: U.S. Census Bureau

Online Research **Data Update** For the latest statistics on marriage and other characteristics of the population, visit www.algebra2.com/data_update to learn more.

Getting Ready for the Next Lesson

PREREQUISITE SKILL **Graph the line with the given equation.**
*(To review **graphing lines**, see Lessons 2-1, 2-2, and 2-3.)*

52. $y = 2x$　　53. $y = -2x$　　54. $y = -\frac{1}{2}x$

55. $y = \frac{1}{2}x$　　56. $y + 2 = 2(x - 1)$　　57. $y + 2 = -2(x - 1)$

8-5 Hyperbolas

What You'll Learn

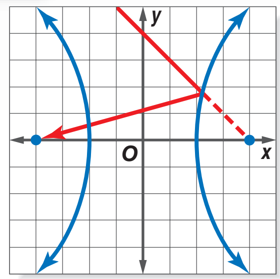
- Write equations of hyperbolas.
- Graph hyperbolas.

Vocabulary

- hyperbola
- foci
- center
- vertex
- asymptote
- transverse axis
- conjugate axis

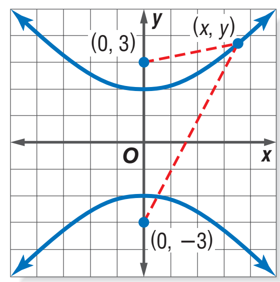
How are hyperbolas different from parabolas?

A hyperbola is a conic section with the property that rays directed toward one focus are reflected toward the other focus. Notice that, unlike the other conic sections, a hyperbola has two branches.



EQUATIONS OF HYPERBOLAS A **hyperbola** is the set of all points in a plane such that the absolute value of the difference of the distances from two fixed points, called the **foci**, is constant.

The hyperbola at the right has foci at (0, 3) and (0, −3). The distances from either of the y-intercepts to the foci are 1 unit and 5 units, so the difference of the distances from any point with coordinates (x, y) on the hyperbola to the foci is 4 or −4 units, depending on the order in which you subtract.



You can use the Distance Formula and the definition of a hyperbola to find an equation of this hyperbola.

$$\underbrace{\text{The distance between } (x, y) \text{ and } (0, 3)} - \underbrace{\text{the distance between } (x, y) \text{ and } (0, -3)} = \pm 4.$$

$$\sqrt{x^2 + (y-3)^2} - \sqrt{x^2 + (y+3)^2} = \pm 4$$

$$\sqrt{x^2 + (y-3)^2} = \pm 4 + \sqrt{x^2 + (y+3)^2} \quad \text{Isolate the radicals.}$$

$$x^2 + (y-3)^2 = 16 \pm 8\sqrt{x^2 + (y+3)^2} + x^2 + (y+3)^2 \quad \text{Square each side.}$$

$$x^2 + y^2 - 6y + 9 = 16 \pm 8\sqrt{x^2 + (y+3)^2} + x^2 + y^2 + 6y + 9$$

$$-12y - 16 = \pm 8\sqrt{x^2 + (y+3)^2} \quad \text{Simplify.}$$

$$3y + 4 = \pm 2\sqrt{x^2 + (y+3)^2} \quad \text{Divide each side by } -4.$$

$$9y^2 + 24y + 16 = 4[x^2 + (y+3)^2] \quad \text{Square each side.}$$

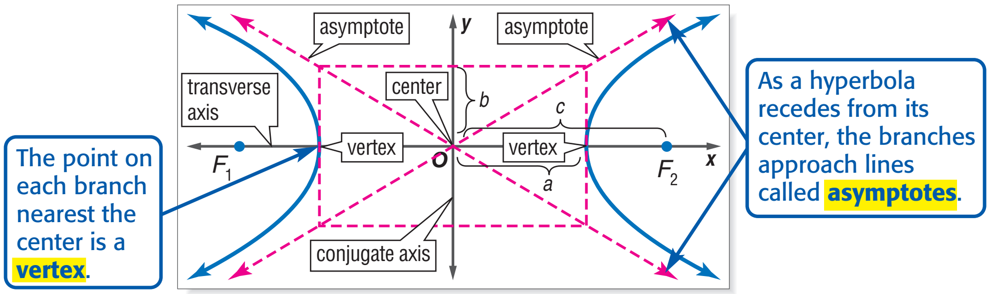
$$9y^2 + 24y + 16 = 4x^2 + 4y^2 + 24y + 36 \quad \text{Distributive Property}$$

$$5y^2 - 4x^2 = 20 \quad \text{Simplify.}$$

$$\frac{y^2}{4} - \frac{x^2}{5} = 1 \quad \text{Divide each side by 20.}$$

An equation of this hyperbola is $\frac{y^2}{4} - \frac{x^2}{5} = 1$.

The diagram below shows the parts of a hyperbola.



A hyperbola has some similarities to an ellipse. The distance from the **center** to a vertex is a units. The distance from the center to a focus is c units. There are two axes of symmetry. The **transverse axis** is a segment of length $2a$ whose endpoints are the vertices of the hyperbola. The **conjugate axis** is a segment of length $2b$ units that is perpendicular to the transverse axis at the center. The values of a, b, and c are related differently for a hyperbola than for an ellipse. For a hyperbola, $c^2 = a^2 + b^2$. The table below summarizes many of the properties of hyperbolas with centers at the origin.

Study Tip

Reading Math

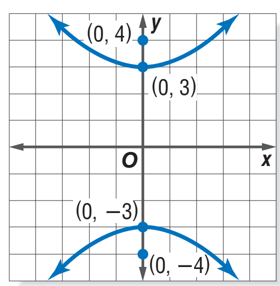
In the standard form of a hyperbola, the squared terms are subtracted (−). For an ellipse, they are added (+).

Key Concept: Equations of Hyperbolas with Centers at the Origin

Standard Form of Equation	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$
Direction of Transverse Axis	horizontal	vertical
Foci	$(c, 0), (-c, 0)$	$(0, c), (0, -c)$
Vertices	$(a, 0), (-a, 0)$	$(0, a), (0, -a)$
Length of Transverse Axis	$2a$ units	$2a$ units
Length of Conjugate Axis	$2b$ units	$2b$ units
Equations of Asymptotes	$y = \pm\frac{b}{a}x$	$y = \pm\frac{a}{b}x$

Example 1 Write an Equation for a Graph

Write an equation for the hyperbola shown at the right.



The center is the midpoint of the segment connecting the vertices, or (0, 0).

The value of a is the distance from the center to a vertex, or 3 units. The value of c is the distance from the center to a focus, or 4 units.

$c^2 = a^2 + b^2$ Equation relating a, b, and c for a hyperbola

$4^2 = 3^2 + b^2$ $c = 4$, $a = 3$

$16 = 9 + b^2$ Evaluate the squares.

$7 = b^2$ Solve for b^2.

Since the transverse axis is vertical, the equation is of the form $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$.

Substitute the values for a^2 and b^2. An equation of the hyperbola is $\frac{y^2}{9} - \frac{x^2}{7} = 1$.

Example 2 Write an Equation Given the Foci and Transverse Axis

NAVIGATION **The LORAN navigational system is based on hyperbolas. Two stations send out signals at the same time. A ship notes the difference in the times at which it receives the signals. The ship is on a hyperbola with the stations at the foci. Suppose a ship determines that the difference of its distances from two stations is 50 nautical miles. The stations are 100 nautical miles apart. Write an equation for a hyperbola on which the ship lies if the stations are at (−50, 0) and (50, 0).**

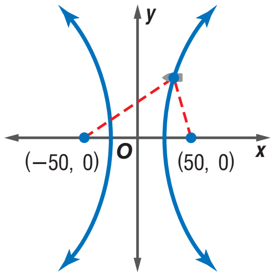
More About. . .

Navigation

LORAN stands for *Long Range Navigation*. The LORAN system is generally accurate to within 0.25 nautical mile.

Source: U.S. Coast Guard

First, draw a figure. By studying either of the x-intercepts, you can see that the difference of the distances from any point on the hyperbola to the stations at the foci is the same as the length of the transverse axis, or $2a$. Therefore, $2a = 50$, or $a = 25$. According to the coordinates of the foci, $c = 50$.



Use the values of a and c to determine the value of b for this hyperbola.

$c^2 = a^2 + b^2$ Equation relating a, b, and c for a hyperbola

$50^2 = 25^2 + b^2$ $c = 50$, $a = 25$

$2500 = 625 + b^2$ Evaluate the squares.

$1875 = b^2$ Solve for b^2.

Since the transverse axis is horizontal, the equation is of the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Substitute the values for a^2 and b^2. An equation of the hyperbola is $\frac{x^2}{625} - \frac{y^2}{1875} = 1$.

GRAPH HYPERBOLAS So far, you have studied hyperbolas that are centered at the origin. A hyperbola may be translated so that its center is at (h, k). This corresponds to replacing x by $x - h$ and y by $y - k$ in both the equation of the hyperbola and the equations of the asymptotes.

Key Concept Equations of Hyperbolas with Centers at (h, k)

Standard Form of Equation	$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$	$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$
Direction of Transverse Axis	horizontal	vertical
Equations of Asymptotes	$y - k = \pm\frac{b}{a}(x - h)$	$y - k = \pm\frac{a}{b}(x - h)$

It is easier to graph a hyperbola if the asymptotes are drawn first. To graph the asymptotes, use the values of a and b to draw a rectangle with dimensions $2a$ and $2b$. The diagonals of the rectangle should intersect at the center of the hyperbola. The asymptotes will contain the diagonals of the rectangle.

Example 3 Graph an Equation in Standard Form

Find the coordinates of the vertices and foci and the equations of the asymptotes for the hyperbola with equation $\frac{x^2}{9} - \frac{y^2}{4} = 1$. Then graph the hyperbola.

The center of this hyperbola is at the origin. According to the equation, $a^2 = 9$ and $b^2 = 4$, so $a = 3$ and $b = 2$. The coordinates of the vertices are (3, 0) and (−3, 0).

(continued on the next page)

$c^2 = a^2 + b^2$ Equation relating a, b, and c for a hyperbola

$c^2 = 3^2 + 2^2$ $a = 3$, $b = 2$

$c^2 = 13$ Simplify.

$c = \sqrt{13}$ Take the square root of each side.

The foci are at $(\sqrt{13}, 0)$ and $(-\sqrt{13}, 0)$.

The equations of the asymptotes are $y = \pm\frac{b}{a}x$ or $y = \pm\frac{2}{3}x$.

You can use a calculator to find some approximate nonnegative values for x and y that satisfy the equation. Since the hyperbola is centered at the origin, it is symmetric about the y-axis. Therefore, the points at $(-8, 4.9)$, $(-7, 4.2)$, $(-6, 3.5)$, $(-5, 2.7)$, $(-4, 1.8)$, and $(-3, 0)$ lie on the graph.

x	y
3	0
4	1.8
5	2.7
6	3.5
7	4.2
8	4.9

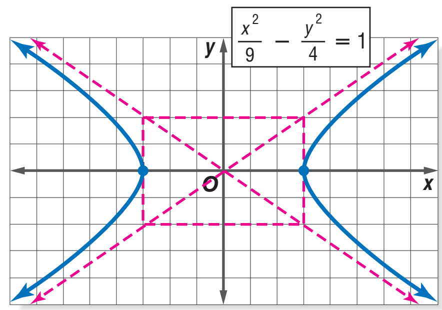
The hyperbola is also symmetric about the x-axis, so the points at $(-8, -4.9)$, $(-7, -4.2)$, $(-6, -3.5)$, $(-5, -2.7)$, $(-4, -1.8)$, $(4, -1.8)$, $(5, -2.7)$, $(6, -3.5)$, $(7, -4.2)$, and $(8, -4.9)$ also lie on the graph.

Study Tip

Graphing Calculator

You can graph a hyperbola on a graphing calculator. Similar to an ellipse, first solve the equation for y. Then graph the two equations that result on the same screen.

Draw a 6-unit by 4-unit rectangle. The asymptotes contain the diagonals of the rectangle. Graph the vertices, which, in this case, are the x-intercepts. Use the asymptotes as a guide to draw the hyperbola that passes through the vertices and the other points. The graph does not intersect the asymptotes.



When graphing a hyperbola given an equation that is not in standard form, begin by rewriting the equation in standard form.

Example 4 Graph an Equation Not in Standard Form

Find the coordinates of the vertices and foci and the equations of the asymptotes for the hyperbola with equation $4x^2 - 9y^2 - 32x - 18y + 19 = 0$. Then graph the hyperbola.

Complete the square for each variable to write this equation in standard form.

$4x^2 - 9y^2 - 32x - 18y + 19 = 0$ Original equation

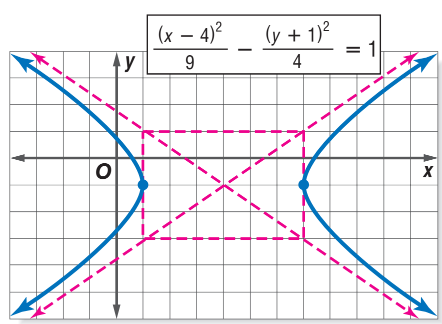
$4(x^2 - 8x + \blacksquare) - 9(y^2 + 2y + \blacksquare) = -19 + 4(\blacksquare) - 9(\blacksquare)$ Complete the squares.

$4(x^2 - 8x + 16) - 9(y^2 + 2y + 1) = -19 + 4(16) - 9(1)$

$4(x - 4)^2 - 9(y + 1)^2 = 36$ Write the trinomials as perfect squares.

$\frac{(x-4)^2}{9} - \frac{(y+1)^2}{4} = 1$ Divide each side by 36.

The graph of this hyperbola is the graph from Example 3 translated 4 units to the right and down 1 unit. The vertices are at $(7, -1)$ and $(1, -1)$, and the foci are at $(4 + \sqrt{13}, -1)$ and $(4 - \sqrt{13}, -1)$. The equations of the asymptotes are $y + 1 = \pm\frac{2}{3}(x - 4)$.



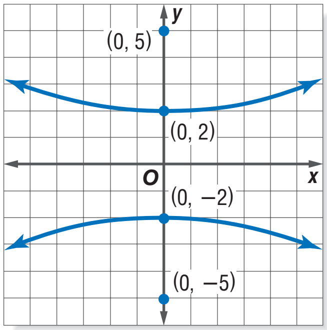
Check for Understanding

Concept Check

1. **Determine** whether the statement is *sometimes*, *always*, or *never* true.
The graph of a hyperbola is symmetric about the x-axis.
2. **Describe** how the graph of $y^2 - \frac{x^2}{k^2} = 1$ changes as k increases.
3. **OPEN ENDED** Find a counterexample to the following statement.
If the equation of a hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, *then* $a^2 \geq b^2$.

Guided Practice

4. Write an equation for the hyperbola shown at the right.
5. A hyperbola has foci at (4, 0) and (−4, 0). The value of a is 1. Write an equation for the hyperbola.



Find the coordinates of the vertices and foci and the equations of the asymptotes for the hyperbola with the given equation. Then graph the hyperbola.

6. $\frac{y^2}{18} - \frac{x^2}{20} = 1$
7. $\frac{(y+6)^2}{20} - \frac{(x-1)^2}{25} = 1$
8. $x^2 - 36y^2 = 36$
9. $5x^2 - 4y^2 - 40x - 16y - 36 = 0$

Application

10. **ASTRONOMY** Comets that pass by Earth only once may follow hyperbolic paths. Suppose a comet's path is modeled by a branch of the hyperbola with equation $\frac{y^2}{225} - \frac{x^2}{400} = 1$. Find the coordinates of the vertices and foci and the equations of the asymptotes for the hyperbola. Then graph the hyperbola.

Practice and Apply

Homework Help

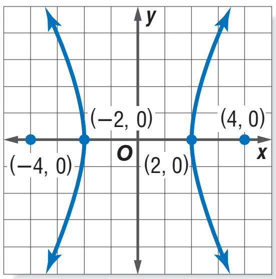
For Exercises	See Examples
11–20, 35	1, 2
21–34, 36–38	3, 4

Extra Practice

See page 846.

Write an equation for each hyperbola.

11.



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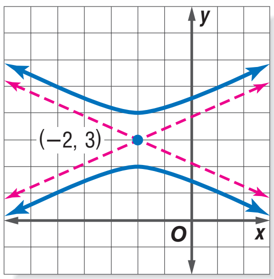

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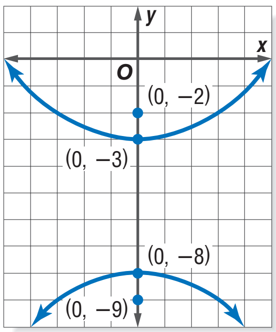

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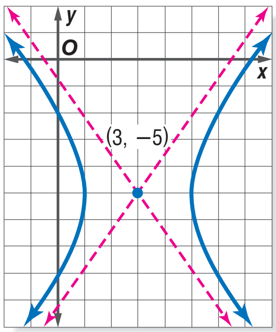


Write an equation for the hyperbola that satisfies each set of conditions.

15. vertices $(-5, 0)$ and $(5, 0)$, conjugate axis of length 12 units

16. vertices $(0, -4)$ and $(0, 4)$, conjugate axis of length 14 units

17. vertices $(9, -3)$ and $(-5, -3)$, foci $(2 \pm \sqrt{53}, -3)$

18. vertices $(-4, 1)$ and $(-4, 9)$, foci $(-4, 5 \pm \sqrt{97})$

19. Find an equation for a hyperbola centered at the origin with a horizontal transverse axis of length 8 units and a conjugate axis of length 6 units.

20. What is an equation for the hyperbola centered at the origin with a vertical transverse axis of length 12 units and a conjugate axis of length 4 units?

Find the coordinates of the vertices and foci and the equations of the asymptotes for the hyperbola with the given equation. Then graph the hyperbola.

21. $\frac{x^2}{81} - \frac{y^2}{49} = 1$

22. $\frac{y^2}{36} - \frac{x^2}{4} = 1$

23. $\frac{y^2}{16} - \frac{x^2}{25} = 1$

24. $\frac{x^2}{9} - \frac{y^2}{25} = 1$

25. $x^2 - 2y^2 = 2$

26. $x^2 - y^2 = 4$

27. $y^2 = 36 + 4x^2$

28. $6y^2 = 2x^2 + 12$

29. $\frac{(y-4)^2}{16} - \frac{(x+2)^2}{9} = 1$

30. $\frac{(y-3)^2}{25} - \frac{(x-2)^2}{16} = 1$

31. $\frac{(x+1)^2}{4} - \frac{(y+3)^2}{9} = 1$

32. $\frac{(x+6)^2}{36} - \frac{(y+3)^2}{9} = 1$

33. $y^2 - 3x^2 + 6y + 6x - 18 = 0$

34. $4x^2 - 25y^2 - 8x - 96 = 0$

FORESTRY **For Exercises 35 and 36, use the following information.**
A forester at an outpost and another forester at the primary station both heard an explosion. The outpost and the primary station are 6 kilometers apart.

35. If one forester heard the explosion 6 seconds before the other, write an equation that describes all the possible locations of the explosion. Place the two forester stations on the x-axis with the midpoint between the stations at the origin. The transverse axis is horizontal. (*Hint*: The speed of sound is about 0.35 kilometer per second.)

36. Draw a sketch of the possible locations of the explosion. Include the ranger stations in the drawing.

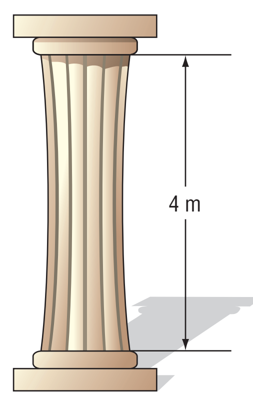
Career Choices

Forester

Foresters work for private companies or governments to protect and manage forest land. They also supervise the planting of trees and use controlled burning to clear weeds, brush, and logging debris.

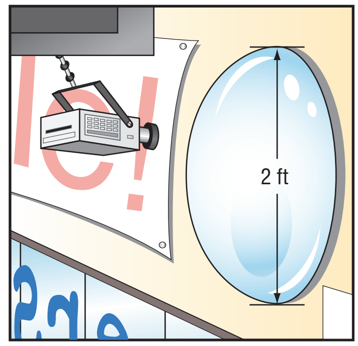
Online Research
For information about a career as a forester, visit: www.algebra2.com/careers

37. **STRUCTURAL DESIGN** An architect's design for a building includes some large pillars with cross sections in the shape of hyperbolas. The curves can be modeled by the equation $\frac{x^2}{0.25} - \frac{y^2}{9} = 1$, where the units are in meters. If the pillars are 4 meters tall, find the width of the top of each pillar and the width of each pillar at the narrowest point in the middle. Round to the nearest centimeter.



38. **CRITICAL THINKING** A hyperbola with a horizontal transverse axis contains the point at $(4, 3)$. The equations of the asymptotes are $y - x = 1$ and $y + x = 5$. Write the equation for the hyperbola.

39. PHOTOGRAPHY A curved mirror is placed in a store for a wide-angle view of the room. The right-hand branch of $\frac{x^2}{1} - \frac{y^2}{3} = 1$ models the curvature of the mirror. A small security camera is placed so that all of the 2-foot diameter of the mirror is visible. If the back of the room lies on $x = -18$, what width of the back of the room is visible to the camera? (*Hint*: Find the equations of the lines through the focus and each edge of the mirror.)



40. WRITING IN MATH Answer the question that was posed at the beginning of the lesson.

How are hyperbolas different from parabolas?

Include the following in your answer:

- differences in the graphs of hyperbolas and parabolas, and
- differences in the reflective properties of hyperbolas and parabolas.

Standardized Test Practice

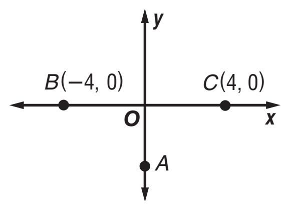
41. A leg of an isosceles right triangle has a length of 5 units. What is the length of the hypotenuse?

(A) $\frac{5\sqrt{2}}{2}$ units (B) 5 units (C) $5\sqrt{2}$ units (D) 10 units

42. In the figure, what is the sum of the slopes of $\overline{AB}$ and $\overline{AC}$?

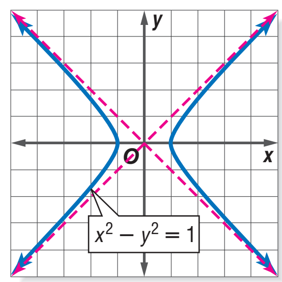
(A) -1 (B) 0

(C) 1 (D) 8



Extending the Lesson

A hyperbola with asymptotes that are not perpendicular is called a **nonrectangular hyperbola**. Most of the hyperbolas you have studied so far are nonrectangular. A **rectangular hyperbola** has perpendicular asymptotes. For example, the graph of $x^2 - y^2 = 1$ is a rectangular hyperbola. The graphs of equations of the form $xy = c$, where c is a constant, are rectangular hyperbolas with the coordinate axes as their asymptotes.



For Exercises 43 and 44, consider the equation $xy = 2$.

43. Plot some points and use them to graph the equation. Be sure to consider negative values for the variables.

44. Find the coordinates of the vertices of the graph of the equation.

45. Graph $xy = -2$.

46. Describe the transformations that can be applied to the graph of $xy = 2$ to obtain the graph of $xy = -2$.

Maintain Your Skills

Mixed Review

Write an equation for the ellipse that satisfies each set of conditions. *(Lesson 8-4)*

47. endpoints of major axis at (1, 2) and (9, 2), endpoints of minor axis at (5, 1) and (5, 3)

48. major axis 8 units long and parallel to y-axis, minor axis 6 units long, center at $(-3, 1)$

49. foci at (5, 4) and $(-3, 4)$, major axis 10 units long

50. Find the center and radius of the circle with equation $x^2 + y^2 - 10x + 2y + 22 = 0$. Then graph the circle. *(Lesson 8-3)*

Solve each equation by factoring. *(Lesson 6-2)*

51. $x^2 + 6x + 8 = 0$

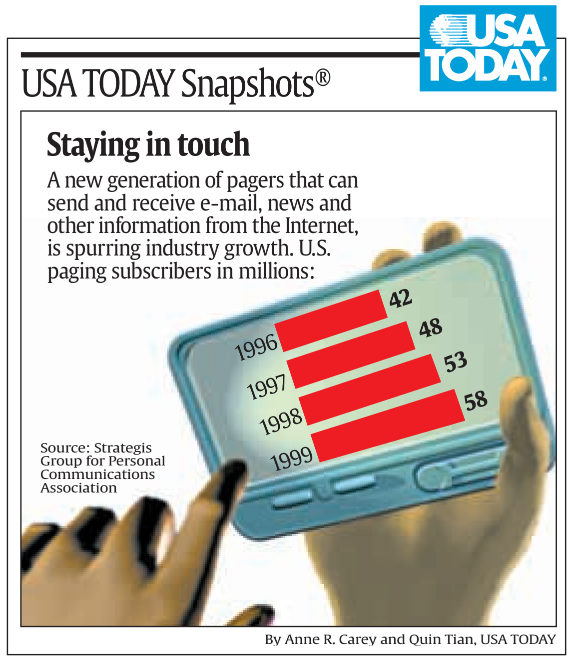
52. $2q^2 + 11q = 21$

Perform the indicated operations, if possible. *(Lesson 4-5)*

53. $\begin{bmatrix} 2 & -1 \\ 0 & 5 \end{bmatrix} \cdot \begin{bmatrix} -3 & 2 \\ 1 & 4 \end{bmatrix}$

54. $[1 \quad -3] \cdot \begin{bmatrix} 4 & -2 & 1 \\ -3 & 2 & 0 \end{bmatrix}$

55. **PAGERS** Refer to the graph at the right. What was the average rate of change of the number of pager subscribers from 1996 to 1999? *(Lesson 2-3)*



56. Solve $|2x + 1| = 9$. *(Lesson 1-4)*

57. Simplify $7x + 8y + 9y - 5x$. *(Lesson 1-2)*

Getting Ready for the Next Lesson

PREREQUISITE SKILL **Each equation is of the form $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$. Identify the values of A, B, and C.** *(To review **coefficients**, see Lesson 5-1.)*

58. $2x^2 + 3xy - 5y^2 = 0$

59. $x^2 - 2xy + 9y^2 = 0$

60. $-3x^2 + xy + 2y^2 + 4x - 7y = 0$

61. $5x^2 - 2y^2 + 5x - y = 0$

62. $x^2 - 4x + 5y + 2 = 0$

63. $xy - 2x - 3y + 6 = 0$

Practice Quiz 2

Lessons 8-4 and 8-5

1. Write an equation of the ellipse with foci at (3, 8) and (3, −6) and endpoints of the major axis at (3, −8) and (3, 10). *(Lesson 8-4)*

Find the coordinates of the center and foci and the lengths of the major and minor axes of the ellipse with the given equation. Then graph the ellipse. *(Lesson 8-4)*

2. $\frac{(x-4)^2}{9} + \frac{(y+2)^2}{1} = 1$

3. $16x^2 + 5y^2 + 32x - 10y - 59 = 0$

Write an equation for the hyperbola that satisfies each set of conditions. *(Lesson 8-5)*

4. vertices (−3, 0) and (3, 0), conjugate axis of length 8 units

5. vertices (−2, 2) and (6, 2), foci $(2 \pm \sqrt{21}, 2)$

CONTENTS

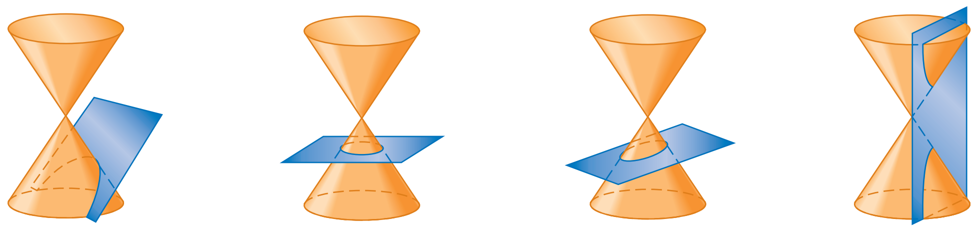
8-6 Conic Sections

What You'll Learn

- Write equations of conic sections in standard form.
- Identify conic sections from their equations.

How can you use a flashlight to make conic sections?

Recall that parabolas, circles, ellipses, and hyperbolas are called conic sections because they are the cross sections formed when a double cone is sliced by a plane. You can use a flashlight and a flat surface to make patterns in the shapes of conic sections.



STANDARD FORM The equation of any conic section can be written in the form of the general quadratic equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, where A, B, and C are not all zero. If you are given an equation in this general form, you can complete the square to write the equation in one of the standard forms you have learned.

Study Tip

Reading Math
In this lesson, the word *ellipse* means an ellipse that is not a circle.

Concept Summary — *Standard Form of Conic Sections*

Conic Section	Standard Form of Equation
Parabola	$y = a(x - h)^2 + k$ or $x = a(y - k)^2 + h$
Circle	$(x - h)^2 + (y - k)^2 = r^2$
Ellipse	$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ or $\frac{(y-k)^2}{a^2} + \frac{(x-h)^2}{b^2} = 1$, $a \neq b$
Hyperbola	$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$ or $\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$

Example 1 Rewrite an Equation of a Conic Section

Write the equation $x^2 + 4y^2 - 6x - 7 = 0$ in standard form. State whether the graph of the equation is a *parabola*, *circle*, *ellipse*, or *hyperbola*. Then graph the equation.

Write the equation in standard form.

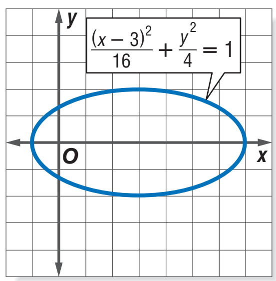
$x^2 + 4y^2 - 6x - 7 = 0$ — Original equation

$x^2 - 6x + \blacksquare + 4y^2 = 7 + \blacksquare$ — Isolate terms.

$x^2 - 6x + 9 + 4y^2 = 7 + 9$ — Complete the square.

$(x - 3)^2 + 4y^2 = 16$ — $x^2 - 6x + 9 = (x - 3)^2$

$\frac{(x-3)^2}{16} + \frac{y^2}{4} = 1$ — Divide each side by 16.



The graph of the equation is an ellipse with its center at (3, 0).

www.algebra2.com/extra_examples

IDENTIFY CONIC SECTIONS Instead of writing the equation in standard form, you can determine what type of conic section an equation of the form $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, where $B = 0$, represents by looking at A and C.

Concept Summary — Identifying Conic Sections

Conic Section	Relationship of *A* and *C*
Parabola	$A = 0$ or $C = 0$, but not both.
Circle	$A = C$
Ellipse	A and C have the same sign and $A \neq C$.
Hyperbola	A and C have opposite signs.

Example 2 Analyze an Equation of a Conic Section

Without writing the equation in standard form, state whether the graph of each equation is a *parabola*, *circle*, *ellipse*, or *hyperbola*.

a. $y^2 - 2x^2 - 4x - 4y - 4 = 0$

$A = -2$ and $C = 1$. Since A and C have opposite signs, the graph is a hyperbola.

b. $4x^2 + 4y^2 + 20x - 12y + 30 = 0$

$A = 4$ and $C = 4$. Since $A = C$, the graph is a circle.

c. $y^2 - 3x + 6y + 12 = 0$

$C = 1$. Since there is no x^2 term, $A = 0$. The graph is a parabola.

Check for Understanding

Concept Check

1. **OPEN ENDED** Write an equation of the form $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, where $A = 2$, that represents a circle.
2. **Write** the general quadratic equation for which $A = 2$, $B = 0$, $C = 0$, $D = -4$, $E = 7$, and $F = 1$.
3. **Explain** why the graph of $x^2 + y^2 - 4x + 2y + 5 = 0$ is a single point.

Guided Practice

Write each equation in standard form. State whether the graph of the equation is a *parabola*, *circle*, *ellipse*, or *hyperbola*. Then graph the equation.

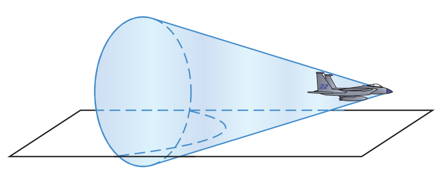
4. $y = x^2 + 3x + 1$
5. $y^2 - 2x^2 - 16 = 0$
6. $x^2 + y^2 = x + 2$
7. $x^2 + 4y^2 + 2x - 24y + 33 = 0$

Without writing the equation in standard form, state whether the graph of each equation is a *parabola*, *circle*, *ellipse*, or *hyperbola*.

8. $y^2 - x - 10y + 34 = 0$
9. $3x^2 + 2y^2 + 12x - 28y + 104 = 0$

Application

AVIATION **For Exercises 10 and 11, use the following information.**
When an airplane flies faster than the speed of sound, it produces a shock wave in the shape of a cone. Suppose the shock wave intersects the ground in a curve that can be modeled by $x^2 - 14x + 4 = 9y^2 - 36y$.



10. Identify the shape of the curve.
11. Graph the equation.

Practice and Apply

Homework Help

For Exercises	See Examples
12–32	1
33–43	2

Extra Practice

See page 846.

Write each equation in standard form. State whether the graph of the equation is a *parabola*, *circle*, *ellipse*, or *hyperbola*. Then graph the equation.

12. $6x^2 + 6y^2 = 162$

13. $4x^2 + 2y^2 = 8$

14. $x^2 = 8y$

15. $4y^2 - x^2 + 4 = 0$

16. $(x - 1)^2 - 9(y - 4)^2 = 36$

17. $y + 4 = (x - 2)^2$

18. $(y - 4)^2 = 9(x - 4)$

19. $x^2 + y^2 + 4x - 6y = -4$

20. $x^2 + y^2 + 6y + 13 = 40$

21. $x^2 - y^2 + 8x = 16$

22. $x^2 + 2y^2 = 2x + 8$

23. $x^2 - 8y + y^2 + 11 = 0$

24. $9y^2 + 18y = 25x^2 + 216$

25. $3x^2 + 4y^2 + 8y = 8$

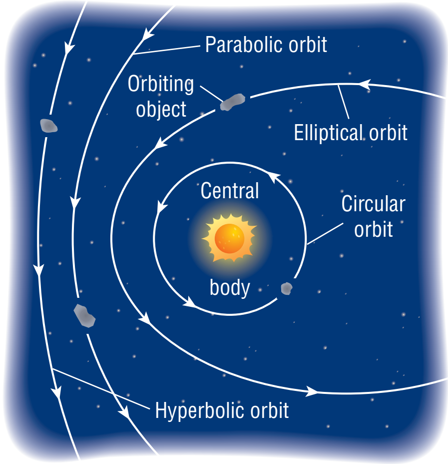
26. $x^2 + 4y^2 - 11 = 2(4y - x)$

27. $y + x^2 = -(8x + 23)$

28. $6x^2 - 24x - 5y^2 - 10y - 11 = 0$

29. $25y^2 + 9x^2 - 50y - 54x = 119$

30. ASTRONOMY The orbits of comets follow paths in the shapes of conic sections. For example, Halley's Comet follows an elliptical orbit with the Sun located at one focus. What type(s) of orbit(s) pass by the Sun only once?



WATER **For Exercises 31 and 32, use the following information.**
If two stones are thrown into a lake at different points, the points of intersection of the resulting ripples will follow a conic section. Suppose the conic section has the equation $x^2 - 2y^2 - 2x - 5 = 0$.

31. Identify the shape of the curve.

32. Graph the equation.

More About. . .

Astronomy

Halley's Comet orbits the Sun about once every 76 years. It will return next in 2061.

Source: www.solarviews.com

Without writing the equation in standard form, state whether the graph of each equation is a *parabola*, *circle*, *ellipse*, or *hyperbola*.

33. $x^2 + y^2 - 8x - 6y + 5 = 0$

34. $3x^2 - 2y^2 + 32y - 134 = 0$

35. $y^2 + 18y - 2x = -84$

36. $7x^2 - 28x + 4y^2 + 8y = -4$

37. $5x^2 + 6x - 4y = x^2 - y^2 - 2x$

38. $2x^2 + 12x + 18 - y^2 = 3(2 - y^2) + 4y$

39. Identify the shape of the graph of the equation $2x^2 + 3x - 4y + 2 = 0$.

40. What type of conic section is represented by the equation $y^2 - 6y = x^2 - 8$?

For Exercises 41–43, match each equation below with the situation that it could represent.

a. $9x^2 + 4y^2 - 36 = 0$

b. $0.004x^2 - x + y - 3 = 0$

c. $x^2 + y^2 - 20x + 30y - 75 = 0$

41. SPORTS the flight of a baseball

42. PHOTOGRAPHY the oval opening in a picture frame

43. GEOGRAPHY the set of all points that are 20 miles from a landmark

CRITICAL THINKING **For Exercises 44 and 45, use the following information.**

The graph of an equation of the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$ is a special case of a hyperbola.

44. Identify the graph of such an equation.

45. Explain how to obtain such a set of points by slicing a double cone with a plane.

46. *WRITING IN MATH* Answer the question that was posed at the beginning of the lesson.

How can you use a flashlight to make conic sections?

Include the following in your answer:

- an explanation of how you could point the flashlight at a ceiling or wall to make a circle, and
- an explanation of how you could point the flashlight to make a branch of a hyperbola.

Standardized Test Practice

47. Which conic section is not symmetric about the y-axis?

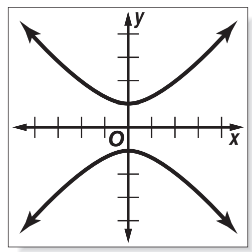
(A) $x^2 - y + 3 = 0$ (B) $y^2 - x^2 - 1 = 0$

(C) $6x^2 + y^2 - 6 = 0$ (D) $x^2 + y^2 - 2x - 3 = 0$

48. What is the equation of the graph at the right?

(A) $y = x^2 + 1$ (B) $y - x = 1$

(C) $y^2 - x^2 = 1$ (D) $x^2 + y^2 = 1$



Extending the Lesson

49. Refer to Exercise 43 on page 440. Eccentricity can be studied for conic sections other than ellipses. The expression for the eccentricity of a hyperbola is $\frac{c}{a}$, just as for an ellipse. The eccentricity of a parabola is 1. Find inequalities for the eccentricities of noncircular ellipses and hyperbolas, respectively.

Maintain Your Skills

Mixed Review

Write an equation of the hyperbola that satisfies each set of conditions. *(Lesson 8-5)*

50. vertices (5, 10) and (5, −2), conjugate axis of length 8 units

51. vertices (6, −6) and (0, −6), foci $(3 \pm \sqrt{13}, -6)$

52. Find the coordinates of the center and foci and the lengths of the major and minor axes of the ellipse with equation $4x^2 + 9y^2 - 24x + 72y + 144 = 0$. Then graph the ellipse. *(Lesson 8-4)*

Simplify. Assume that no variable equals 0. *(Lesson 5-1)*

53. $(x^3)^4$ **54.** $(m^5n^{-3})^2m^2n^7$ **55.** $\frac{x^2y^{-3}}{x^{-5}y}$

56. **HEALTH** The prediction equation $y = 205 - 0.5x$ relates a person's maximum heart rate for exercise y and age x. Use the equation to find the maximum heart rate for an 18-year-old. *(Lesson 2-5)*

Getting Ready for the Next Lesson

PREREQUISITE SKILL **Solve each system of equations.**

(To review ***solving systems of linear equations****, see Lesson 3-2.)*

57. $y = x + 4$
$2x + y = 10$

58. $4x + y = 14$
$4x - y = 10$

59. $x + 5y = 10$
$3x - 2y = -4$

Algebra Activity

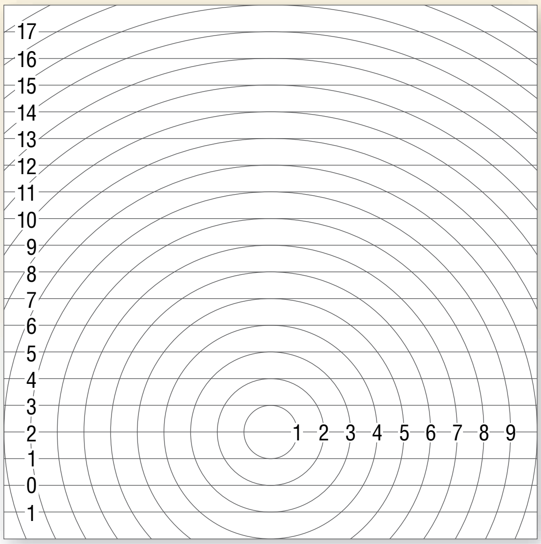
A Follow-Up of Lesson 8-6

Conic Sections

Recall that a parabola is the set of all points that are equidistant from the focus and the directrix.

You can draw a parabola based on this definition by using special conic graph paper. This graph paper contains a series of concentric circles equally spaced from each other and a series of parallel lines tangent to the circles.

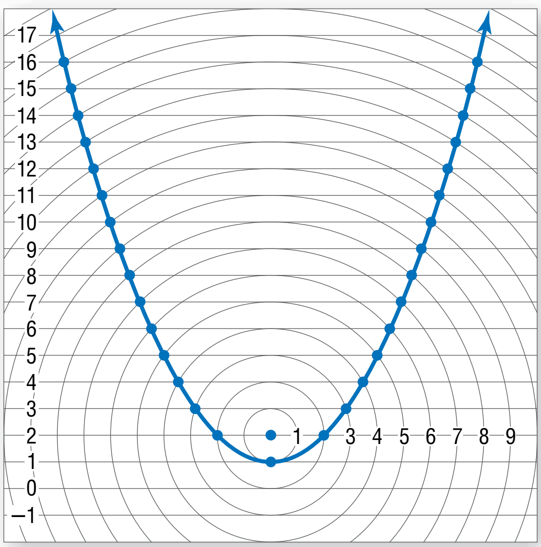
Number the circles consecutively beginning with the smallest circle. Number the lines with consecutive integers as shown in the sample at the right. Be sure that line 1 is tangent to circle 1.



Activity 1

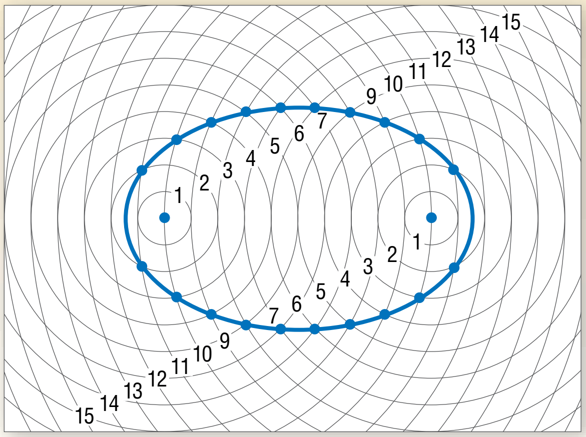
Mark the point at the intersection of circle 1 and line 1. Mark both points that are on line 2 and circle 2. Continue this process, marking both points on line 3 and circle 3, and so on. Then connect the points with a smooth curve.

Look at the diagram at the right. What shape is the graph? Note that every point on the graph is equidistant from the center of the small circle and the line labeled 0. The center of the small circle is the focus of the parabola, and line 0 is the directrix.



Activity 2

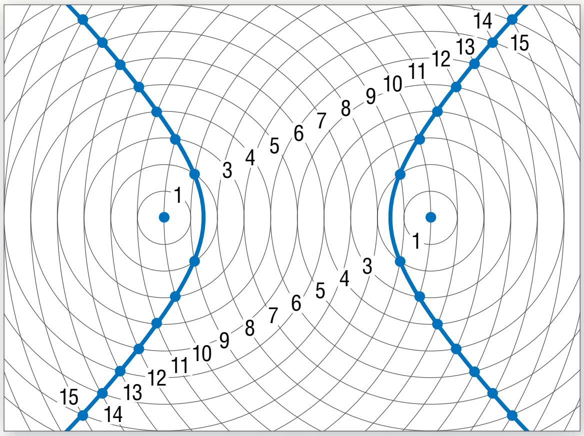
An ellipse is the set of points such that the sum of the distances from two fixed points is constant. The two fixed points are called the foci.



- Use graph paper like that shown. It contains two small circles and a series of concentric circles from each. The concentric circles are tangent to each other as shown.
- Choose the constant 13. Mark the points at the intersections of circle 9 and circle 4, because $9 + 4 = 13$. Continue this process until you have marked the intersection of all circles whose sum is 13.
- Connect the points to form a smooth curve. The curve is an ellipse whose foci are the centers of the two small circles on the graph paper.

Activity 3

A hyperbola is the set of points such that the difference of the distances from two fixed points is constant. The two fixed points are called the foci.



- Use the same type of graph paper that you used for the ellipse in Activity 2. Choose the constant 7. Mark the points at the intersections of circle 9 and circle 2, because $9 - 2 = 7$. Continue this process until you have marked the intersections of all circles whose difference in radius is 7.
- Connect the points to form a hyperbola.

Model and Analyze

1. Use the type of graph paper you used in Activity 1. Mark the intersection of line 0 and circle 2. Then mark the two points on line 1 and circle 3, the two points on line 2 and circle 4, and so on. Draw the new parabola. Continue this process and make as many parabolas as you can on one sheet of the graph paper. The focus is always the center of the small circle. Why are the resulting graphs parabolas?

2. In Activity 2, you drew an ellipse such that the sum of the distances from two fixed points was 13. Choose 10, 11, 12, 14, and so on for that sum, and draw as many ellipses as you can on one piece of the graph paper.

a. Why can you not start with 9 as the sum?

b. What happens as the sum increases? decreases?

3. In Activity 3, you drew a hyperbola such that the difference of the distances from two fixed points was 7. Choose other numbers and draw as many hyperbolas as you can on one piece of graph paper. What happens as the difference increases? decreases?

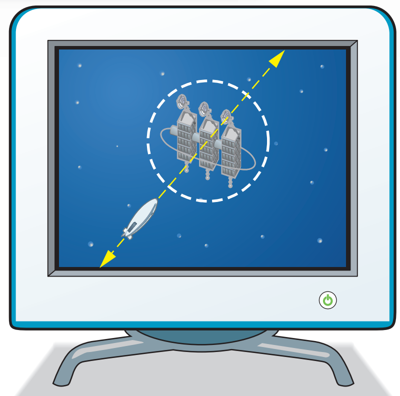
8-7 Solving Quadratic Systems

What You'll Learn

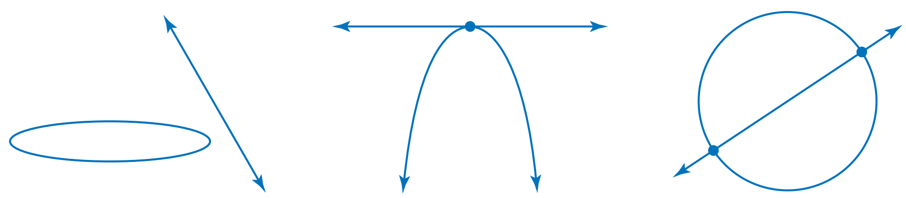
- Solve systems of quadratic equations algebraically and graphically.
- Solve systems of quadratic inequalities graphically.

How do systems of equations apply to video games?

Computer software often uses a coordinate system to keep track of the locations of objects on the screen. Suppose an enemy space station is located at the center of the screen, which is the origin in a coordinate system. The space station is surrounded by a circular force field of radius 50 units. If the spaceship you control is flying toward the center along the line with equation $y = 3x$, the point where the ship hits the force field is a solution of a system of equations.



SYSTEMS OF QUADRATIC EQUATIONS If the graphs of a system of equations are a conic section and a line, the system may have zero, one, or two solutions. Some of the possible situations are shown below.



You have solved systems of linear equations graphically and algebraically. You can use similar methods to solve systems involving quadratic equations.

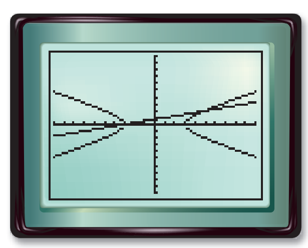
Example 1 Linear-Quadratic System

Solve the system of equations.

$x^2 - 4y^2 = 9$
$4y - x = 3$

You can use a graphing calculator to help visualize the relationships of the graphs of the equations and predict the number of solutions.

Solve each equation for y to obtain $y = \pm\frac{\sqrt{x^2 - 9}}{2}$ and $y = \frac{1}{4}x + \frac{3}{4}$. Enter the functions $y = \frac{\sqrt{x^2 - 9}}{2}$, $y = -\frac{\sqrt{x^2 - 9}}{2}$, and $y = \frac{1}{4}x + \frac{3}{4}$ on the Y= screen. The graph indicates that the hyperbola and line intersect in two points. So the system has two solutions.



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(continued on the next page)

Use substitution to solve the system. First rewrite $4y - x = 3$ as $x = 4y - 3$.

$x^2 - 4y^2 = 9$ First equation in the system

$(4y - 3)^2 - 4y^2 = 9$ Substitute $4y - 3$ for x.

$12y^2 - 24y = 0$ Simplify.

$y^2 - 2y = 0$ Divide each side by 12.

$y(y - 2) = 0$ Factor.

$y = 0$ or $y - 2 = 0$ Zero Product Property

$y = 2$ Solve for y.

Now solve for x.

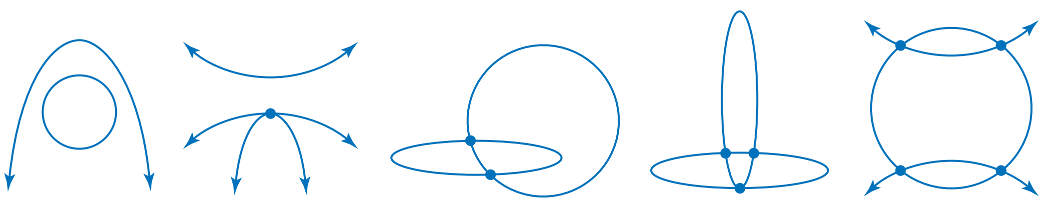
$x = 4y - 3$ $\quad x = 4y - 3$ Equation for x in terms of y

$= 4(0) - 3$ $\quad = 4(2) - 3$ Substitute the y values.

$= -3$ $\quad = 5$ Simplify.

The solutions of the system are $(-3, 0)$ and $(5, 2)$. Based on the graph, these solutions are reasonable.

If the graphs of a system of equations are two conic sections, the system may have zero, one, two, three, or four solutions. Some of the possible situations are shown below.

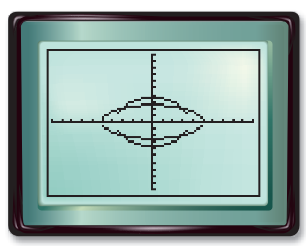


Example 2 Quadratic-Quadratic System

Solve the system of equations.

$y^2 = 13 - x^2$
$x^2 + 4y^2 = 25$

A graphing calculator indicates that the circle and ellipse intersect in four points. So, this system has four solutions.



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Use the elimination method to solve the system.

$y^2 = 13 - x^2$
$x^2 + 4y^2 = 25$

$-x^2 - y^2 = -13$ Rewrite the first original equation.

$(+)\ x^2 + 4y^2 = 25$ Second original equation

$3y^2 = 12$ Add.

$y^2 = 4$ Divide each side by 3.

$y = \pm 2$ Take the square root of each side.

Study Tip

Graphing Calculators

If you use ZSquare on the ZOOM menu, the graph of the first equation will look like a circle.

Substitute 2 and -2 for y in either of the original equations and solve for x.

$x^2 + 4y^2 = 25$	$x^2 + 4y^2 = 25$	Second original equation
$x^2 + 4(2)^2 = 25$	$x^2 + 4(-2)^2 = 25$	Substitute for y.
$x^2 = 9$	$x^2 = 9$	Subtract 16 from each side.
$x = \pm 3$	$x = \pm 3$	Take the square root of each side.

The solutions are (3, 2), (−3, 2), (−3, −2), and (3, −2).

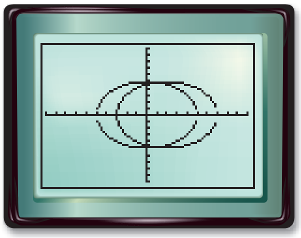
A graphing calculator can be used to approximate the solutions of a system of equations.

Graphing Calculator Investigation

Quadratic Systems

The calculator screen shows the graphs of two circles.

Think and Discuss



[−10, 10] scl: 1 by [−10, 10] scl: 1

1. Write the system of equations represented by the graph.
2. Enter the equations into a TI-83 Plus and use the intersect feature on the CALC menu to solve the system. Round to the nearest hundredth.
3. Solve the system algebraically.
4. Can you always find the exact solution of a system using a graphing calculator? Explain.

Use a graphing calculator to solve each system of equations. Round to the nearest hundredth.

5. $y = x + 2$
   $x^2 + y^2 = 9$

6. $3x^2 + y^2 = 11$
   $y = x^2 + x + 1$

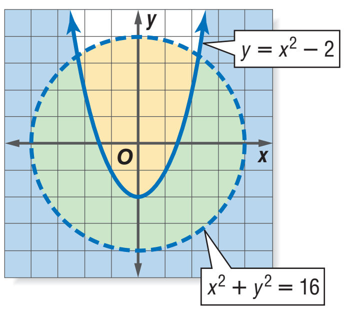
SYSTEMS OF QUADRATIC INEQUALITIES You have learned how to solve systems of linear inequalities by graphing. Systems of quadratic inequalities are also solved by graphing.

The graph of an inequality involving a parabola, circle, or ellipse is either the interior or the exterior of the conic section. The graph of an inequality involving a hyperbola is either the region between the branches or the two regions inside the branches. As with linear inequalities, examine the inequality symbol to determine whether to include the boundary.

Example 3 System of Quadratic Inequalities

Solve the system of inequalities by graphing.

$y \leq x^2 - 2$
$x^2 + y^2 < 16$



The graph of $y \leq x^2 - 2$ is the parabola $y = x^2 - 2$ and the region outside or below it. This region is shaded blue.

The graph of $x^2 + y^2 < 16$ is the interior of the circle $x^2 + y^2 = 16$. This region is shaded yellow.

The intersection of these regions, shaded green, represents the solution of the system of inequalities.

Study Tip

Graphing Quadratic Inequalities

If you are unsure about which region to shade, you can test one or more points, as you did with linear inequalities.

Check for Understanding

Concept Check

1. **Graph** each system of equations. Use the graph to solve the system.
   a. $4x - 3y = 0$
   $x^2 + y^2 = 25$
   b. $y = 5 - x^2$
   $y = 2x^2 + 2$
2. **Sketch** a parabola and an ellipse that intersect at exactly three points.
3. **OPEN ENDED** Write a system of quadratic equations for which (2, 6) is a solution.

Guided Practice

Find the exact solution(s) of each system of equations.

4. $y = 5$
   $y^2 = x^2 + 9$
5. $y - x = 1$
   $x^2 + y^2 = 25$
6. $3x = 8y^2$
   $8y^2 - 2x^2 = 16$
7. $5x^2 + y^2 = 30$
   $9x^2 - y^2 = -16$

Solve each system of inequalities by graphing.

8. $x + y < 4$
   $9x^2 - 4y^2 \geq 36$
9. $x^2 + y^2 < 25$
   $4x^2 - 9y^2 < 36$

Application

10. **EARTHQUAKES** In a coordinate system where a unit represents one mile, the epicenter of an earthquake was determined to be 50 miles from a station at the origin. It was also 40 miles from a station at (0, 30) and 13 miles from a station at (35, 18). Where was the epicenter located?

Practice and Apply

Homework Help

For Exercises	See Examples
11–16, 25, 29	1
17–24, 26–28, 30, 31	2
32–37	3

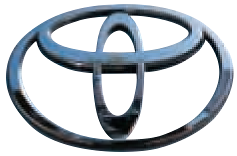
Extra Practice

See page 847.

Find the exact solution(s) of each system of equations.

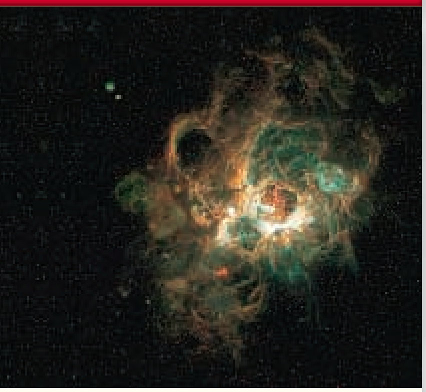
11. $y = x + 2$
    $y = x^2$
12. $y = x + 3$
    $y = 2x^2$
13. $x^2 + y^2 = 36$
    $y = x + 2$
14. $y^2 + x^2 = 9$
    $y = 7 - x$
15. $\frac{x^2}{30} + \frac{y^2}{6} = 1$
    $x = y$
16. $\frac{x^2}{36} - \frac{y^2}{4} = 1$
    $x = y$
17. $4x + y^2 = 20$
    $4x^2 + y^2 = 100$
18. $y + x^2 = 3$
    $x^2 + 4y^2 = 36$
19. $x^2 + y^2 = 64$
    $x^2 + 64y^2 = 64$
20. $y^2 + x^2 = 25$
    $y^2 + 9x^2 = 25$
21. $y^2 = x^2 - 25$
    $x^2 - y^2 = 7$
22. $y^2 = x^2 - 7$
    $x^2 + y^2 = 25$
23. $2x^2 + 8y^2 + 8x - 48y + 30 = 0$
    $2x^2 - 8y^2 = -48y + 90$
24. $3x^2 - 20y^2 - 12x + 80y - 96 = 0$
    $3x^2 + 20y^2 = 80y + 48$
25. Where do the graphs of the equations $y = 2x + 1$ and $2x^2 + y^2 = 11$ intersect?
26. What are the coordinates of the points that lie on the graphs of both $x^2 + y^2 = 25$ and $2x^2 + 3y^2 = 66$?
27. **ROCKETS** Two rockets are launched at the same time, but from different heights. The height y in feet of one rocket after t seconds is given by $y = -16t^2 + 150t + 5$. The height of the other rocket is given by $y = -16t^2 + 160t$. After how many seconds are the rockets at the same height?

28. ADVERTISING The corporate logo for an automobile manufacturer is shown at the right. Write a system of three equations to model this logo.



29. MIRRORS A hyperbolic mirror is a mirror in the shape of one branch of a hyperbola. Such a mirror reflects light rays directed at one focus toward the other focus. Suppose a hyperbolic mirror is modeled by the upper branch of the hyperbola with equation $\frac{y^2}{9} - \frac{x^2}{16} = 1$. A light source is located at $(-10, 0)$. Where should the light from the source hit the mirror so that the light will be reflected to $(0, -5)$?

More About. . .



Astronomy

The astronomical unit (AU) is the mean distance between Earth and the Sun. One AU is about 93 million miles or 150 million kilometers.

Source: www.infoplease.com

ASTRONOMY **For Exercises 30 and 31, use the following information.**

The orbit of Pluto can be modeled by the equation $\frac{x^2}{39.5^2} + \frac{y^2}{38.3^2} = 1$, where the units are astronomical units. Suppose a comet is following a path modeled by the equation $x = y^2 + 20$.

30. Find the point(s) of intersection of the orbits of Pluto and the comet. Round to the nearest tenth.

31. Will the comet necessarily hit Pluto? Explain.

Solve each system of inequalities by graphing.

32. $x + 2y > 1$
$x^2 + y^2 \leq 25$

33. $x + y \leq 2$
$4x^2 - y^2 \geq 4$

34. $x^2 + y^2 \geq 4$
$4y^2 + 9x^2 \leq 36$

35. $x^2 + y^2 < 36$
$4x^2 + 9y^2 > 36$

36. $y^2 < x$
$x^2 - 4y^2 < 16$

37. $x^2 \leq y$
$y^2 - x^2 \geq 4$

CRITICAL THINKING **For Exercises 38–42, find all values of k for which the system of equations has the given number of solutions. If no values of k meet the condition, write *none*.**

$$x^2 + y^2 = k^2 \qquad \frac{x^2}{9} + \frac{y^2}{4} = 1$$

38. no solutions

39. one solution

40. two solutions

41. three solutions

42. four solutions

43. *WRITING IN MATH* Answer the question that was posed at the beginning of the lesson.

How do systems of equations apply to video games?

Include the following in your answer:

- a linear-quadratic system of equations that applies to this situation,
- an explanation of how you know that the spaceship is headed directly toward the center of the screen, and
- the coordinates of the point at which the spaceship will hit the force field, assuming that the spaceship moves from the bottom of the screen toward the center.

Standardized Test Practice

A B C D

44. If $\boxed{x}$ is defined to be $x^2 - 4x$ for all numbers x, which of the following is the greatest?

(A) $\boxed{0}$ (B) $\boxed{1}$ (C) $\boxed{2}$ (D) $\boxed{3}$

45. How many three-digit numbers are divisible by 3?

(A) 299 (B) 300 (C) 301 (D) 302

Graphing Calculator

SYSTEMS OF EQUATIONS **Write a system of equations that satisfies each condition. Use a graphing calculator to verify that you are correct.**

46. two parabolas that intersect in two points

47. a hyperbola and a circle that intersect in three points

48. a circle and an ellipse that do not intersect

49. a circle and an ellipse that intersect in four points

50. a hyperbola and an ellipse that intersect in two points

51. two circles that intersect in three points

Maintain Your Skills

Mixed Review

Write each equation in standard form. State whether the graph of the equation is a *parabola*, *circle*, *ellipse*, or *hyperbola*. Then graph the equation. *(Lesson 8-6)*

52. $x^2 + y^2 + 4x + 2y - 6 = 0$

53. $9x^2 + 4y^2 - 24y = 0$

54. Find the coordinates of the vertices and foci and the equations of the asymptotes of the hyperbola with the equation $6y^2 - 2x^2 = 24$. Then graph the hyperbola. *(Lesson 8-5)*

Solve each equation by factoring. *(Lesson 6-5)*

55. $x^2 + 7x = 0$

56. $x^2 - 3x = 0$

57. $22 = 9x^2 + 4x$

58. $35 = -2x + x^2$

59. $9x^2 + 24 = -16$

60. $8x^2 + 2x = 3$

For Exercises 61 and 62, complete parts a–c for each quadratic equation. *(Lesson 6-5)*

a. Find the value of the discriminant.

b. Describe the number and type of roots.

c. Find the exact solutions by using the Quadratic Formula.

61. $5x^2 = 2$

62. $-3x^2 + 6x - 7 = 0$

Simplify. *(Lesson 5-9)*

63. $(3 + 2i) - (1 - 7i)$

64. $(8 - i)(4 - 3i)$

65. $\frac{2 + 3i}{1 + 2i}$

66. **CHEMISTRY** The mass of a proton is about 1.67×10^{-27} kilogram. The mass of an electron is about 9.11×10^{-31} kilogram. About how many times as massive as an electron is a proton? *(Lesson 5-1)*

Evaluate each determinant. *(Lesson 4-3)*

67. $\begin{vmatrix} 2 & -3 \\ 2 & 0 \end{vmatrix}$

68. $\begin{vmatrix} -4 & -2 \\ 5 & 3 \end{vmatrix}$

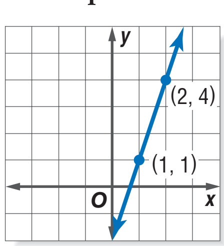
69. $\begin{vmatrix} 2 & 1 & -2 \\ 4 & 0 & 3 \\ -3 & 1 & 7 \end{vmatrix}$

70. Solve the system of equations. *(Lesson 3-5)*

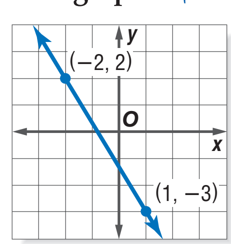
$r + s + t = 15$
$r + t = 12$
$s + t = 10$

Write an equation in slope-intercept form for each graph. *(Lesson 2-4)*

71.



72.



Chapter 8

Study Guide and Review

Vocabulary and Concept Check

asymptote (p. 442)
center of a circle (p. 426)
center of a hyperbola (p. 442)
center of an ellipse (p. 434)
circle (p. 426)
conic section (p. 419)
conjugate axis (p. 442)
directrix (p. 419)
Distance Formula (p. 413)
ellipse (p. 433)
foci of a hyperbola (p. 441)
foci of an ellipse (p. 433)
focus of a parabola (p. 419)
hyperbola (p. 441)
latus rectum (p. 421)
major axis (p. 434)
Midpoint Formula (p. 412)
minor axis (p. 434)
parabola (p. 419)
tangent (p. 427)
transverse axis (p. 442)
vertex of a hyperbola (p. 442)

Tell whether each statement is *true* or *false*. If the statement is false, correct it to make it true.

1. An ellipse is the set of all points in a plane such that the sum of the distances from two given points in the plane, called the foci, is constant.
2. The major axis is the longer of the two axes of symmetry of an ellipse.
3. The formula used to find the distance between two points in a coordinate plane is $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.
4. A parabola is the set of all points that are the same distance from a given point called the directrix and a given line called the focus.
5. The radius is the distance from the center of a circle to any point on the circle.
6. The conjugate axis of a hyperbola is a line segment parallel to the transverse axis.
7. A conic section is formed by slicing a double cone by a plane.
8. A hyperbola is the set of all points in a plane such that the absolute value of the sum of the distances from any point on the hyperbola to two given points is constant.
9. The midpoint formula is given by $\left(\frac{x_1 - x_2}{2}, \frac{y_1 - y_2}{2}\right)$.
10. The set of all points in a plane that are equidistant from a given point in a plane, called the center, forms a circle.

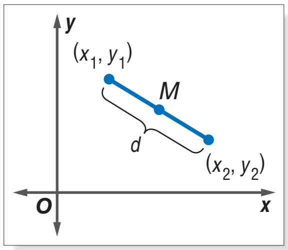
Lesson-by-Lesson Review

8-1 Midpoint and Distance Formulas

See pages 412–416.

Concept Summary

- Midpoint Formula: $M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$
- Distance Formula: $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$



Examples

1 **Find the midpoint of a segment whose endpoints are at (−5, 9) and (11, −1).**

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) = \left(\frac{-5 + 11}{2}, \frac{9 + (-1)}{2}\right)$$ Let $(x_1, y_1) = (-5, 9)$ and $(x_2, y_2) = (11, -1)$.

$$= \left(\frac{6}{2}, \frac{8}{2}\right) \text{ or } (3, 4)$$ Simplify.

2 **Find the distance between $P(6, -4)$ and $Q(-3, 8)$.**

$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ Distance Formula

$= \sqrt{(-3 - 6)^2 + [8 - (-4)]^2}$ Let $(x_1, y_1) = (6, -4)$ and $(x_2, y_2) = (-3, 8)$.

$= \sqrt{81 + 144}$ Subtract.

$= \sqrt{225}$ or 15 units Simplify.

Exercises **Find the midpoint of the line segment with endpoints at the given coordinates.** *See Example 1 on page 412.*

11. (1, 2), (4, 6) **12.** (−8, 0), (−2, 3) **13.** $\left(\frac{3}{5}, -\frac{7}{4}\right), \left(\frac{1}{4}, -\frac{2}{5}\right)$

Find the distance between each pair of points with the given coordinates. *See Examples 2 and 3 on pages 413 and 414.*

14. (−2, 10), (−2, 13) **15.** (8, 5), (−9, 4) **16.** (7, −3), (1, 2)

8-2 Parabolas

See pages 419–425.

Concept Summary

Parabolas		
Standard Form	$y = a(x - h)^2 + k$	$x = a(y - k)^2 + h$
Vertex	(h, k)	(h, k)
Axis of Symmetry	$x = h$	$y = k$
Focus	$\left(h, k + \frac{1}{4a}\right)$	$\left(h + \frac{1}{4a}, k\right)$
Directrix	$y = k - \frac{1}{4a}$	$x = h - \frac{1}{4a}$

Example **Graph $4y - x^2 = 14x - 27$.**

First write the equation in the form $y = a(x - h)^2 + k$.

$4y - x^2 = 14x - 27$ Original equation

$4y = x^2 + 14x - 27$ Isolate the terms with x.

$4y = (x^2 + 14x + ■) - 27 - ■$ Complete the square.

$4y = (x^2 + 14x + 49) - 27 - 49$ Add and subtract 49, since $\left(\frac{14}{2}\right)^2 = 49$.

$4y = (x + 7)^2 - 76$ $x^2 + 14x + 49 = (x + 7)^2$

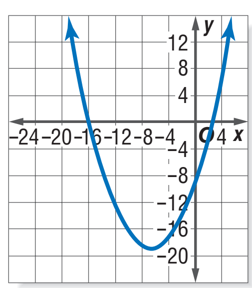
$y = \frac{1}{4}(x + 7)^2 - 19$ Divide each side by 4.

vertex: $(-7, -19)$ axis of symmetry: $x = -7$

focus: $\left(-7, -19 + \frac{1}{4\left(\frac{1}{4}\right)}\right)$ or $(-7, -18)$

directrix: $y = -19 - \frac{1}{4\left(\frac{1}{4}\right)}$ or $y = -20$

direction of opening: upward since $a > 0$



CONTENTS

Exercises **Identify the coordinates of the vertex and focus, the equations of the axis of symmetry and directrix, and the direction of opening of the parabola with the given equation. Then find the length of the latus rectum and graph the parabola.** *See Examples 2–4 on pages 420–423.*

17. $(x - 1)^2 = 12(y - 1)$

18. $y + 6 = 16(x - 3)^2$

19. $x^2 - 8x + 8y + 32 = 0$

20. $x = 16y^2$

21. Write an equation for a parabola with vertex (0, 1) and focus (0, −1). Then graph the parabola. *See Example 4 on pages 422 and 423.*

8-3 Circles

See pages 426–431.

Concept Summary

- The equation of a circle with center (h, k) and radius r can be written in the form $(x - h)^2 + (y - k)^2 = r^2$.

Example **Graph $x^2 + y^2 + 8x - 24y + 16 = 0$.**

First write the equation in the form $(x - h)^2 + (y - k)^2 = r^2$.

$x^2 + y^2 + 8x - 24y + 16 = 0$ Original equation

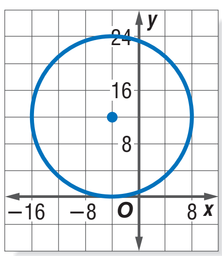
$x^2 + 8x + ■ + y^2 - 24y + ■ = -16 + ■ + ■$ Complete the squares.

$x^2 + 8x + 16 + y^2 - 24y + 144 = -16 + 16 + 144$ $\left(\frac{8}{2}\right)^2 = 16, \left(\frac{-24}{2}\right)^2 = 144$

$(x + 4)^2 + (y - 12)^2 = 144$ Write the trinomials as squares.

The center of the circle is at (−4, 12) and the radius is 12.

Now draw the graph.



Exercises **Write an equation for the circle that satisfies each set of conditions.** *See Example 1 on page 426.*

22. center (2, −3), radius 5 units

23. center (−4, 0), radius $\frac{3}{4}$ unit

24. endpoints of a diameter at (9, 4) and (−3, −2)

25. center at (−1, 2), tangent to x-axis

Find the center and radius of the circle with the given equation. Then graph the circle. *See Examples 4 and 5 on page 428.*

26. $x^2 + y^2 = 169$

27. $(x + 5)^2 + (y - 11)^2 = 49$

28. $x^2 + y^2 - 6x + 16y - 152 = 0$

29. $x^2 + y^2 + 6x - 2y - 15 = 0$

CONTENTS

8-4 Ellipses

See pages 433–440.

Concept Summary

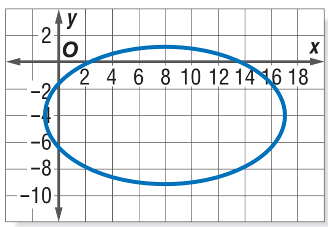
Ellipses		
Standard Form of Equation	$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$	$\frac{(y-k)^2}{a^2} + \frac{(x-h)^2}{b^2} = 1$
Direction of Major Axis	horizontal	vertical

Example **Graph $x^2 + 3y^2 - 16x + 24y + 31 = 0$.**

First write the equation in standard form by completing the squares.

$x^2 + 3y^2 - 16x + 24y + 31 = 0$	Original equation
$x^2 - 16x + \blacksquare + 3(y^2 + 8y + \blacksquare) = -31 + \blacksquare + 3(\blacksquare)$	Complete the squares.
$x^2 - 16x + 64 + 3(y^2 + 8y + 16) = -31 + 64 + 3(16)$	$\left(\frac{-16}{2}\right)^2 = 64$, $\left(\frac{8}{2}\right)^2 = 16$
$(x-8)^2 + 3(y+4)^2 = 81$	Write the trinomials as squares.
$\frac{(x-8)^2}{81} + \frac{(y+4)^2}{27} = 1$	Divide each side by 81.

The center of the ellipse is at (8, −4). The length of the major axis is 18, and the length of the minor axis is $6\sqrt{3}$.



Exercises

30. Write an equation for the ellipse with endpoints of the major axis at (4, 1) and (−6, 1) and endpoints of the minor axis at (−1, 3) and (−1, −1).
See Examples 1 and 2 on pages 434 and 435.

Find the coordinates of the center and foci and the lengths of the major and minor axes for the ellipse with the given equation. Then graph the ellipse.
See Examples 3 and 4 on pages 436 and 437.

31. $\frac{x^2}{16} + \frac{y^2}{25} = 1$ **32.** $\frac{(x+2)^2}{16} + \frac{(y-3)^2}{9} = 1$ **33.** $x^2 + 4y^2 - 2x + 16y + 13 = 0$

8-5 Hyperbolas

See pages 441–448.

Concept Summary

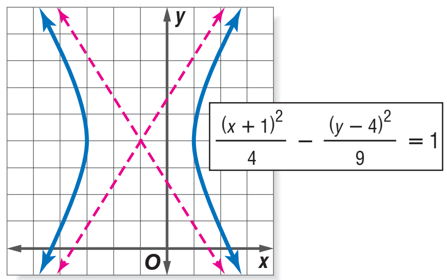
Hyperbolas		
Standard Form	$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$	$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$
Transverse Axis	horizontal	vertical
Asymptotes	$y - k = \pm\frac{b}{a}(x-h)$	$y - k = \pm\frac{a}{b}(x-h)$

Example **Graph $9x^2 - 4y^2 + 18x + 32y - 91 = 0$.**

Complete the square for each variable to write this equation in standard form.

$9x^2 - 4y^2 + 18x + 32y - 91 = 0$	Original equation
$9(x^2 + 2x + \blacksquare) - 4(y^2 - 8y + \blacksquare) = 91 + 9(\blacksquare) - 4(\blacksquare)$	Complete the squares.
$9(x^2 + 2x + 1) - 4(y^2 - 8y + 16) = 91 + 9(1) - 4(16)$	$\left(\frac{2}{2}\right)^2 = 1, \left(\frac{-8}{2}\right)^2 = 16$
$9(x + 1)^2 - 4(y - 4)^2 = 36$	Write the trinomials as squares.
$\frac{(x+1)^2}{4} - \frac{(y-4)^2}{9} = 1$	Divide each side by 36.

The center is at $(-1, 4)$. The vertices are at $(-3, 4)$ and $(1, 4)$ and the foci are at $\left(-1 \pm \sqrt{13}, 4\right)$. The equations of the asymptotes are $y - 4 = \pm\frac{3}{2}(x + 1)$.



Exercises

34. Write an equation for a hyperbola that has vertices at (2, 5) and (2, 1) and a conjugate axis of length 6 units. *See Example 1 on page 442.*

Find the coordinates of the vertices and foci and the equations of the asymptotes for the hyperbola with the given equation. Then graph the hyperbola.
See Examples 3 and 4 on pages 443 and 444.

35. $\frac{y^2}{4} - \frac{x^2}{9} = 1$

36. $\frac{(x-2)^2}{1} - \frac{(y+1)^2}{9} = 1$

37. $9y^2 - 16x^2 = 144$

38. $16x^2 - 25y^2 - 64x - 336 = 0$

8-6 Conic Sections

See pages 449–452.

Concept Summary

- Conic sections can be identified directly from their equations of the form $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, assuming $B = 0$.

Conic Section	Relationship of *A* and *C*
Parabola	$A = 0$ or $C = 0$, but not both.
Circle	$A = C$
Ellipse	*A* and *C* have the same sign and $A \neq C$.
Hyperbola	*A* and *C* have opposite signs.

Example **Without writing the equation in standard form, state whether the graph of $4x^2 + 9y^2 + 16x - 18y - 11 = 0$ is a *parabola*, *circle*, *ellipse*, or *hyperbola*.**

In this equation, $A = 4$ and $C = 9$. Since *A* and *C* are both positive and $A \neq C$, the graph is an ellipse.

Chapter 8 For More ...

- Extra Practice, see pages 845–847.
- Mixed Problem Solving, see page 869.

Exercises **Write each equation in standard form. State whether the graph of the equation is a *parabola*, *circle*, *ellipse*, or *hyperbola*. Then graph the equation.**
See Example 1 on page 449.

39. $x^2 + 4x - y = 0$

40. $9x^2 + 4y^2 = 36$

41. $-4x^2 + y^2 + 8x - 8 = 0$

42. $x^2 + y^2 - 4x - 6y + 4 = 0$

Without writing the equation in standard form, state whether the graph of each equation is a *parabola*, *circle*, *ellipse*, or *hyperbola*. *See Example 2 on page 450.*

43. $7x^2 + 9y^2 = 63$

44. $x^2 - 8x + 16 = 6y$

45. $x^2 + 4x + y^2 - 285 = 0$

46. $5y^2 + 2y + 4x - 13x^2 = 81$

8-7 Solving Quadratic Systems

See pages 455–460.

Concept Summary

- Systems of quadratic equations can be solved using substitution and elimination.
- A system of quadratic equations can have zero, one, two, three, or four solutions.

Example

Solve the system of equations.
$\mathbf{x^2 + y^2 + 2x - 12y + 12 = 0}$
$\mathbf{y + x = 0}$

Use substitution to solve the system.

First, rewrite $y + x = 0$ as $y = -x$.

$x^2 + y^2 + 2x - 12y + 12 = 0$ — First original equation

$x^2 + (-x)^2 + 2x - 12(-x) + 12 = 0$ — Substitute $-x$ for y.

$2x^2 + 14x + 12 = 0$ — Simplify.

$x^2 + 7x + 6 = 0$ — Divide each side by 2.

$(x + 6)(x + 1) = 0$ — Factor.

$x + 6 = 0$ or $x + 1 = 0$ — Zero Product Property.

$x = -6$ $\quad$ $x = -1$ — Solve for x.

Now solve for y.

$y = -x$ $\quad$ $y = -x$ — Equation for y in terms of x

$= -(-6)$ or 6 $\quad$ $= -(-1)$ or 1 — Substitute the x values.

The solutions of the system are $(-6, 6)$ and $(-1, 1)$.

Exercises **Find the exact solution(s) of each system of equations.**
See Examples 1 and 2 on pages 455–457.

47. $x^2 + y^2 - 18x + 24y + 200 = 0$
$4x + 3y = 0$

48. $4x^2 + y^2 = 16$
$x^2 + 2y^2 = 4$

Solve each system of inequalities by graphing. *See Example 3 on page 457.*

49. $y < x$
$y > x^2 - 4$

50. $x^2 + y^2 \le 9$
$x^2 + 4y^2 \le 16$

Chapter 8 Practice Test

Vocabulary and Concepts

Choose the letter that best matches each description.

a. ellipse
b. parabola
c. hyperbola

1. the set of all points in a plane that are the same distance from a given point, the focus, and a given line, the directrix
2. the set of all points in a plane such that the absolute value of the difference of the distances from two fixed points, the foci, is constant
3. the set of all points in a plane such that the sum of the distances from two fixed points, the foci, is constant

Skills and Applications

Find the midpoint of the line segment with endpoints at the given coordinates.

4. $(7, 1), (-5, 9)$
5. $\left(\frac{3}{8}, -1\right), \left(-\frac{8}{5}, 2\right)$
6. $(-13, 0), (-1, -8)$

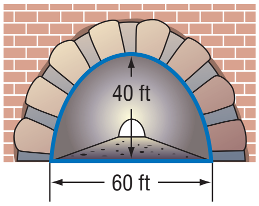
Find the distance between each pair of points with the given coordinates.

7. $(-6, 7), (3, 2)$
8. $\left(\frac{1}{2}, \frac{5}{2}\right), \left(-\frac{3}{4}, -\frac{11}{4}\right)$
9. $(8, -1), (8, -9)$

State whether the graph of each equation is a *parabola*, *circle*, *ellipse*, or *hyperbola*. Then graph the equation.

10. $x^2 + 4y^2 = 25$
11. $y = 4x^2 + 1$
12. $x^2 = 36 - y^2$
13. $(x + 4)^2 = 7(y + 5)$
14. $4x^2 - 26y^2 + 10 = 0$
15. $25x^2 + 49y^2 = 1225$
16. $-(y^2 - 24) = x^2 + 10x$
17. $5x^2 - y^2 = 49$
18. $\frac{1}{3}x^2 - 4 = y$
19. $\frac{y^2}{9} - \frac{x^2}{25} = 1$

20. **TUNNELS** The opening of a tunnel is in the shape of a semielliptical arch. The arch is 60 feet wide and 40 feet high. Find the height of the arch 12 feet from the edge of the tunnel.



Find the exact solution(s) of each system of equations.

21. $x^2 + y^2 = 100$
 $y = 2 - x$
22. $x^2 + 2y^2 = 6$
 $x + y = 1$
23. $x^2 - y^2 - 12x + 12y = 36$
 $x^2 + y^2 - 12x - 12y + 36 = 0$

24. Solve the system of inequalities by graphing.
 $x^2 - y^2 \geq 1$
 $x^2 + y^2 \leq 16$

25. **STANDARDIZED TEST PRACTICE** Which is *not* the equation of a parabola?
 (A) $y = 2x^2 + 4x - 9$
 (B) $3x + 2y^2 + y + 1 = 0$
 (C) $x^2 + 2y^2 + 8y = 8$
 (D) $x = \frac{1}{2}(y - 1)^2 + 5$

www.algebra2.com/chapter_test

CONTENTS

Chapter 8 Standardized Test Practice

Part 1 Multiple Choice

Record your answers on the answer sheet provided by your teacher or on a sheet of paper.

1. The product of a prime number and a composite number must be

(A) prime. (B) composite.
(C) even. (D) negative.

2. In 1990, the population of Clayton was 54,200, and the population of Montrose was 47,500. By 2000, the population of each city had decreased by exactly 5%. How many more people lived in Clayton than in Montrose in 2000?

(A) 335 (B) 5085
(C) 6365 (D) 6700

3. If 4% of n is equal to 40% of p, then n is what percent of $10p$?

(A) $\frac{1}{1000}\%$ (B) 10%
(C) 100% (D) 1,000%

4. Leroy bought m magazines at d dollars per magazine and p paperback books at $2d + 1$ dollars per book. Which of the following represents the total amount Leroy spent?

(A) $d(m + 2p) + p$ (B) $(m + p)(3d + 1)$
(C) $md + 2pd + 1$ (D) $pd(m + 2)$

The Princeton Review **Test-Taking Tip**

Questions 3, 4 In problems with variables, you can substitute values to try to eliminate some of the answer choices. For example, in Question 3, choose a value for n and compute the corresponding value of p. Then find $\frac{n}{10p}$ to answer the question.

5. What is the midpoint of the line segment whose endpoints are at $(-5, -3)$ and $(-1, 4)$?

(A) $\left(-3, -\frac{1}{2}\right)$ (B) $\left(-3, \frac{1}{2}\right)$
(C) $\left(-2, \frac{7}{2}\right)$ (D) $\left(-2, \frac{1}{2}\right)$

6. Point $M(-2, 3)$ is the midpoint of line segment NP. If point N has coordinates $(-7, 1)$, then what are the coordinates of point P?

(A) $(-5, 2)$ (B) $(-4, 6)$
(C) $\left(-\frac{9}{2}, 2\right)$ (D) $(3, 5)$

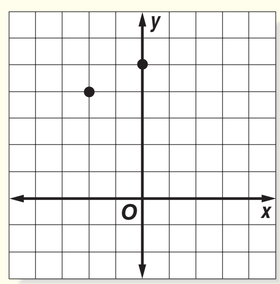
7. Which equation's graph is a parabola?

(A) $3x^2 - 2y^2 = 10$
(B) $4x^2 + 3y^2 = 20$
(C) $2x^2 + 2y^2 = 15$
(D) $3x^2 + 4y = 8$

8. What is the center of the circle with equation $x^2 + y^2 - 4x + 6y - 9 = 0$?

(A) $(-4, 6)$ (B) $(-2, 3)$
(C) $(2, -3)$ (D) $(3, 3)$

9. What is the distance between the points shown in the graph?



(A) $\sqrt{3}$ units
(B) $\sqrt{5}$ units
(C) 3 units
(D) $\sqrt{17}$ units

10. The median of seven test scores is 52, the mode is 64, the lowest score is 40, and the average is 53. If the scores are integers, what is the greatest possible test score?

(A) 68 (B) 72 (C) 76 (D) 84

Part 2 Short Response/Grid In

Record your answers on the answer sheet provided by your teacher or on a sheet of paper.

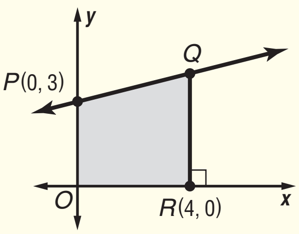
11. What is the least positive integer p for which $2^{2p} + 3$ is not a prime number?

12. The ratio of cars to SUVs in a parking lot is 4 to 5. After 6 cars leave the parking lot, the ratio of cars to SUVs becomes 1 to 2. How many SUVs are in the parking lot?

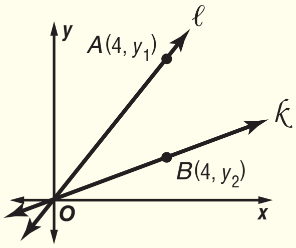
13. Each dimension of a rectangular box is an integer greater than 1. If the area of one side of the box is 27 square units and the area of another side is 12 square units, what is the volume of the box in cubic units?

14. Let the operation * be defined as $a * b = 2ab - (a + b)$. If $4 * x = 10$, then what is the value of x?

15. If the slope of line PQ in the figure is $\frac{1}{4}$, what is the area of quadrilateral $OPQR$?



16. In the figure, the slope of line ℓ is $\frac{5}{4}$, and the slope of line k is $\frac{3}{8}$. What is the distance from point A to point B?



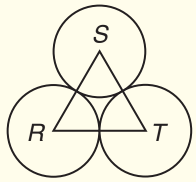
17. If $(2x - 3)(4x + n) = ax^2 + bx - 15$ for all values of x, what is the value of $a + b$?

Part 3 Quantitative Comparison

Compare the quantity in Column A and the quantity in Column B. Then determine whether:

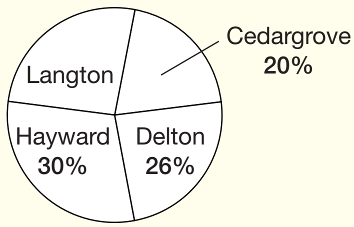
- Ⓐ **the quantity in Column A is greater,**
- Ⓑ **the quantity in Column B is greater,**
- Ⓒ **the two quantities are equal, or**
- Ⓓ **the relationship cannot be determined from the information given.**

	Column A	Column B
18. Tangent circles R, S, and T each have an area of 16π square units.		



	Column A	Column B
18.	perimeter of $\triangle RST$	circumference of $\odot R$
19.	the ratio of girls to boys in Class A that contains 4 more boys than girls	the ratio of girls to boys in Class B that contains 4 more girls than boys

20. Percent of Lakewood School Students Living in Each Town



	Column A	Column B
20.	the number of students who do *not* live in Langton	the number of students who do *not* live in Delton

21. $2 < nk < 10$

n and k are positive integers.

	Column A	Column B
21.	nk	$n + k$

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CONTENTS