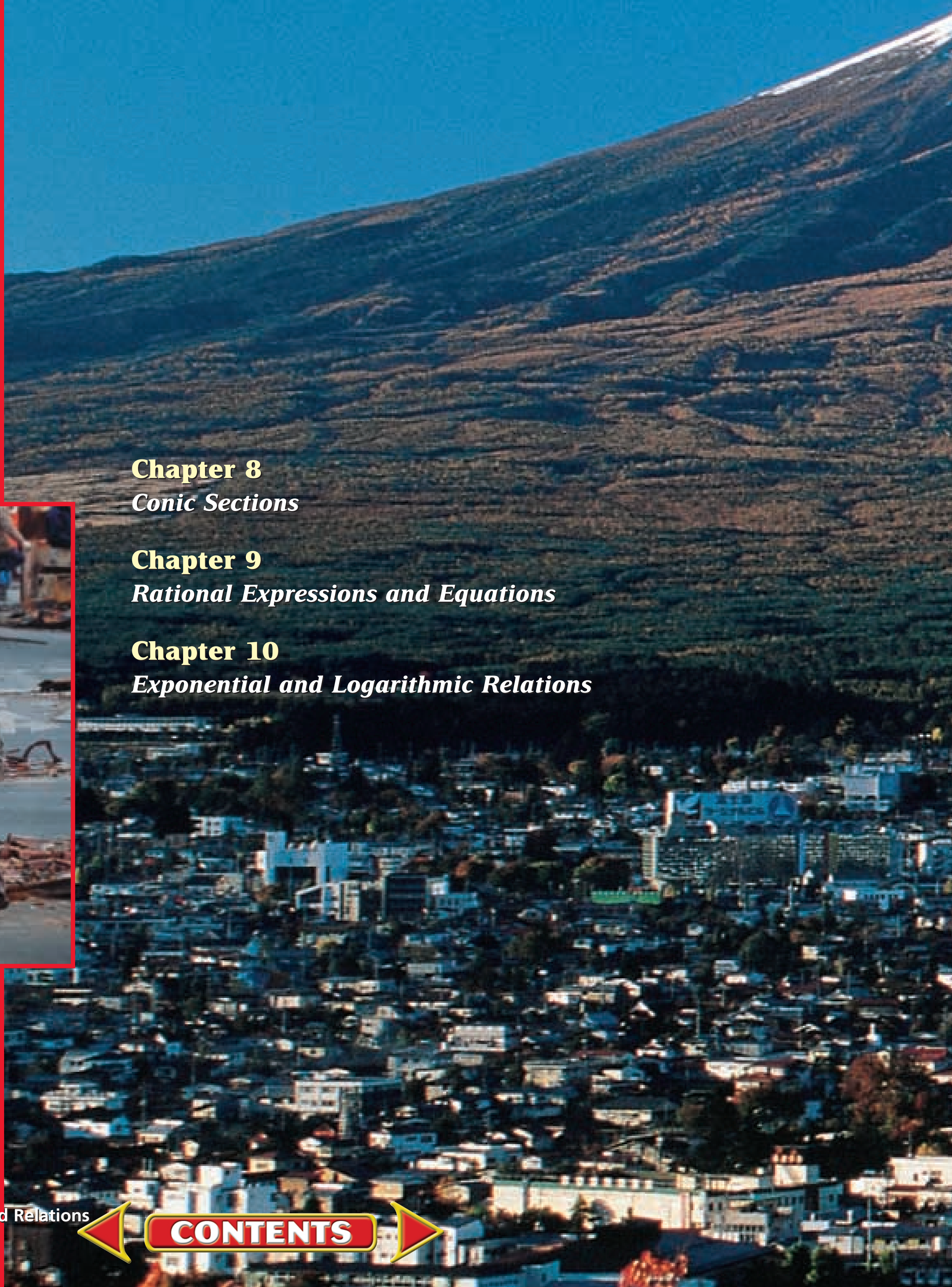
UNIT

3

Advanced Functions and Relations

You can use functions and relations to investigate events like earthquakes. In this unit, you will learn about conic sections, rational expressions and equations, and exponential and logarithmic functions.



WebQuest Internet Project

On Quake Anniversary, Japan Still Worries

Source: *USA TODAY,* January 16, 2001

"As Japan marks the sixth anniversary of the devastating Kobe earthquake this week, a different seismic threat is worrying the country: Mount Fuji. Researchers have measured a sudden increase of small earthquakes on the volcano, indicating there is movement of magma underneath its snowcapped, nearly symmetrical cone about 65 miles from Tokyo." In this project, you will explore how functions and relations are related to locating, measuring, and classifying earthquakes.

Log on to www.algebra2.com/webquest. Begin your WebQuest by reading the Task.

Then continue working on your WebQuest as you study Unit 3.

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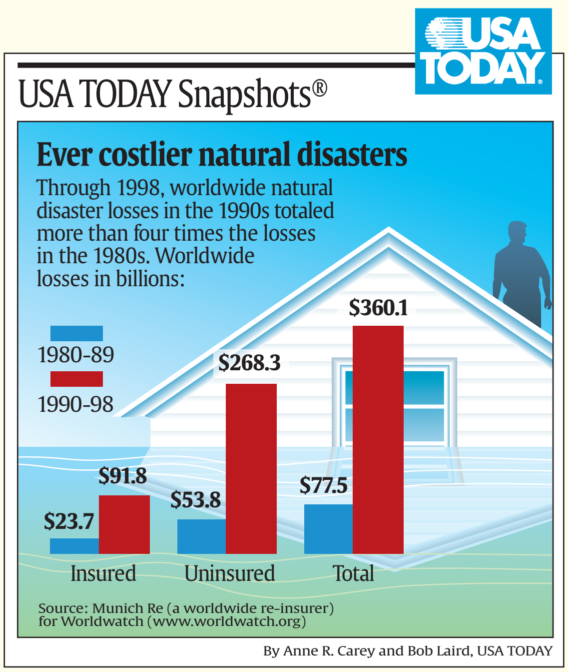


Chapter 8

Conic Sections

What You'll Learn

- **Lesson 8-1** Use the Midpoint and Distance Formulas.
- **Lessons 8-2 through 8-5** Write and graph equations of parabolas, circles, ellipses, and hyperbolas.
- **Lesson 8-6** Identify conic sections.
- **Lesson 8-7** Solve systems of quadratic equations and inequalities.

Key Vocabulary

- **parabola** (p. 419)
- **conic section** (p. 419)
- **circle** (p. 426)
- **ellipse** (p. 433)
- **hyperbola** (p. 441)

Why It's Important

Many planets, comets, and satellites have orbits in curves called *conic sections*. These curves include parabolas, circles, ellipses, and hyperbolas. The Moon's orbit is almost a perfect circle. *You will learn more about the orbits in Lessons 8-2 through 8-7.*

Getting Started

Prerequisite Skills To be successful in this chapter, you'll need to master these skills and be able to apply them in problem-solving situations. Review these skills before beginning Chapter 8.

For Lessons 8-2 through 8-6 — Completing the Square

Solve each equation by completing the square. *(For review, see Lesson 6-4.)*

1. $x^2 + 10x + 24 = 0$ **2.** $x^2 - 2x + 2 = 0$ **3.** $2x^2 + 5x - 12 = 0$

For Lessons 8-2 through 8-6 — Translation Matrices

A translation is given for each figure.

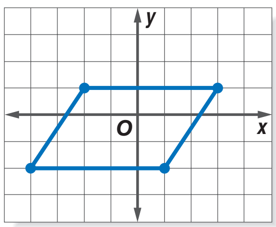
a. Write the vertex matrix for the given figure.

b. Write the translation matrix.

c. Find the coordinates in matrix form of the vertices of the translated figure.

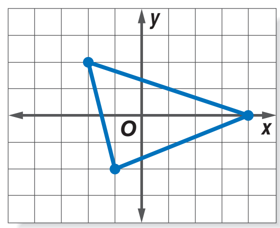
(For review, see Lesson 4-4.)

4.



translated 4 units left and 2 units up

5.



translated 5 units right and 3 units down

For Lesson 8-7 — Graph Linear Inequalities

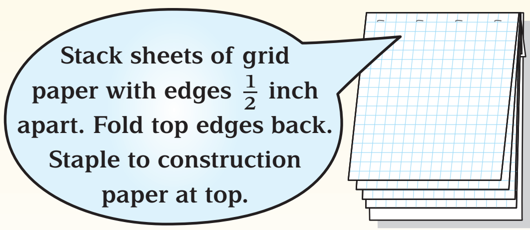
Graph each inequality. *(For review, see Lesson 2-7.)*

6. $y < x + 2$ **7.** $x + y \leq 3$ **8.** $2x - 3y > 6$

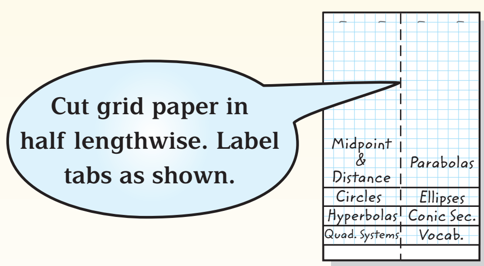
FOLDABLES™ Study Organizer

Make this Foldable to help you organize information about conic sections. Begin with four sheets of grid paper and one piece of construction paper.

Step 1 Fold and Staple



Step 2 Cut and Label



Reading and Writing As you read and study the chapter, use each tab to write notes, formulas, and examples for each conic section.

CONTENTS

8-1 Midpoint and Distance Formulas

What You'll Learn

- Find the midpoint of a segment on the coordinate plane.
- Find the distance between two points on the coordinate plane.

How are the Midpoint and Distance Formulas used in emergency medicine?

A square grid is superimposed on a map of eastern Nebraska where emergency medical assistance by helicopter is available from both Lincoln and Omaha. Each side of a square represents 10 miles. You can use the formulas in this lesson to determine whether the site of an emergency is closer to Lincoln or to Omaha.

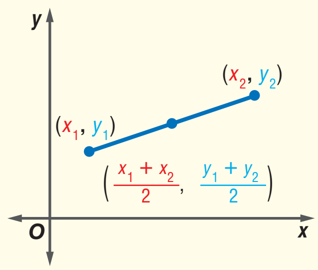
THE MIDPOINT FORMULA Recall that point M is the midpoint of segment PQ if M is between P and Q and $PM = MQ$. There is a formula for the coordinates of the midpoint of a segment in terms of the coordinates of the endpoints. *You will show that this formula is correct in Exercise 41.*

Study Tip

Midpoints
The coordinates of the midpoint are the means of the coordinates of the endpoints.

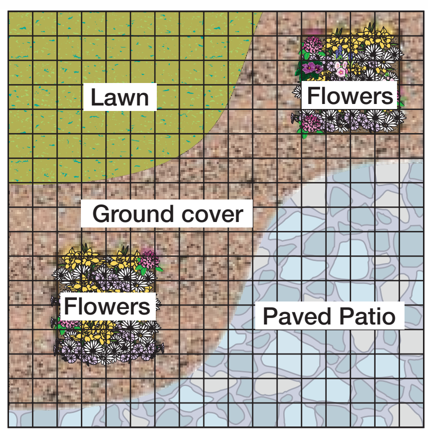
Key Concept — Midpoint Formula

- **Words** If a line segment has endpoints at (x_1, y_1) and (x_2, y_2), then the midpoint of the segment has coordinates $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.
- **Symbols** midpoint $= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$
- **Model**



Example 1 Find a Midpoint

LANDSCAPING A landscape design includes two square flower beds and a sprinkler halfway between them. Find the coordinates of the sprinkler if the origin is at the lower left corner of the grid.



The centers of the flower beds are at (4, 5) and (14, 13). The sprinkler will be at the midpoint of the segment joining these points.

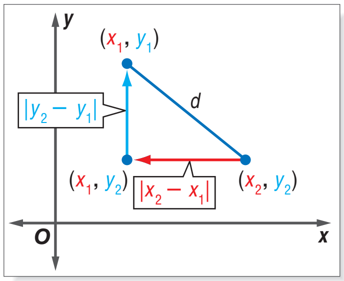
$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) = \left(\frac{4 + 14}{2}, \frac{5 + 13}{2}\right)$$

$$= \left(\frac{18}{2}, \frac{18}{2}\right) \text{ or } (9, 9)$$

The sprinkler will have coordinates (9, 9).

THE DISTANCE FORMULA Recall that the distance between two points on a number line whose coordinates are a and b is $|a - b|$ or $|b - a|$. You can use this fact and the Pythagorean Theorem to derive a formula for the distance between two points on a coordinate plane.

Suppose (x_1, y_1) and (x_2, y_2) name two points. Draw a right triangle with vertices at these points and the point (x_1, y_2). The lengths of the legs of the right triangle are $|x_2 - x_1|$ and $|y_2 - y_1|$. Let d represent the distance between (x_1, y_1) and (x_2, y_2). Now use the Pythagorean Theorem.



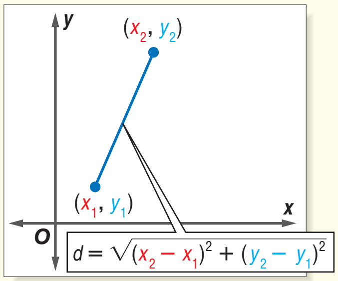
$c^2 = a^2 + b^2$	Pythagorean Theorem
$d^2 = \|x_2 - x_1\|^2 + \|y_2 - y_1\|^2$	Replace c with d, a with $\|x_2 - x_1\|$, and b with $\|y_2 - y_1\|$.
$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$	$\|x_2 - x_1\|^2 = (x_2 - x_1)^2$; $\|y_2 - y_1\|^2 = (y_2 - y_1)^2$
$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$	Find the nonnegative square root of each side.

Study Tip

Distance
In mathematics, distances are always positive.

Key Concept — Distance Formula

- **Words** The distance between two points with coordinates (x_1, y_1) and (x_2, y_2) is given by $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.
- **Model**



Example 2 Find the Distance Between Two Points

What is the distance between $A(-3, 6)$ and $B(4, -4)$?

$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$	Distance Formula
$= \sqrt{[4 - (-3)]^2 + (-4 - 6)^2}$	Let $(x_1, y_1) = (-3, 6)$ and $(x_2, y_2) = (4, -4)$.
$= \sqrt{7^2 + (-10)^2}$	Subtract.
$= \sqrt{49 + 100}$ or $\sqrt{149}$	Simplify.

The distance between the points is $\sqrt{149}$ units.

Standardized Test Practice

Example 3 Find the Farthest Point

Multiple-Choice Test Item

Which point is farthest from $(-1, 3)$?

(A) (2, 4) (B) (−4, 1) (C) (0, 5) (D) (3, −2)

Read the Test Item

The word *farthest* refers to the greatest distance.

(continued on the next page)

The Princeton Review

Test-Taking Tip

If you forget the Distance Formula, you can draw a right triangle and use the Pythagorean Theorem, as shown on the previous page.

Solve the Test Item

Use the Distance Formula to find the distance from (−1, 3) to each point.

Distance to (2, 4)

$$d = \sqrt{[2-(-1)]^2 + (4-3)^2}$$
$$= \sqrt{3^2 + 1^2} \text{ or } \sqrt{10}$$

Distance to (−4, 1)

$$d = \sqrt{[-4-(-1)]^2 + (1-3)^2}$$
$$= \sqrt{(-3)^2 + (-2)^2} \text{ or } \sqrt{13}$$

Distance to (0, 5)

$$d = \sqrt{[0-(-1)]^2 + (5-3)^2}$$
$$= \sqrt{1^2 + 2^2} \text{ or } \sqrt{5}$$

Distance to (3, −2)

$$d = \sqrt{[3-(-1)]^2 + (-2-3)^2}$$
$$= \sqrt{4^2 + (-5)^2} \text{ or } \sqrt{41}$$

The greatest distance is $\sqrt{41}$ units. So, the farthest point from (−1, 3) is (3, −2). The answer is D.

Check for Understanding

Concept Check

1. **Explain** how you can determine in which quadrant the midpoint of the segment with endpoints at (−6, 8) and (4, 3) lies without actually calculating the coordinates.
2. **Identify** all of the points that are equidistant from the endpoints of a given segment.
3. **OPEN ENDED** Find two points that are $\sqrt{29}$ units apart.

Guided Practice

Find the midpoint of each line segment with endpoints at the given coordinates.

4. (−5, 6), (1, 7)
5. (8, 9), (−3, −4.5)

Find the distance between each pair of points with the given coordinates.

6. (2, −4), (10, −10)
7. (7, 8), (−4, 9)
8. (0.5, 1.4), (1.1, 2.9)

Standardized Test Practice

9. Which of the following points is closest to (2, −4)?

   (A) (3, 1) (B) (−2, 0) (C) (1, 5) (D) (4, −2)

Practice and Apply

Homework Help

For Exercises	See Examples
10–23	1
24–40	2, 3

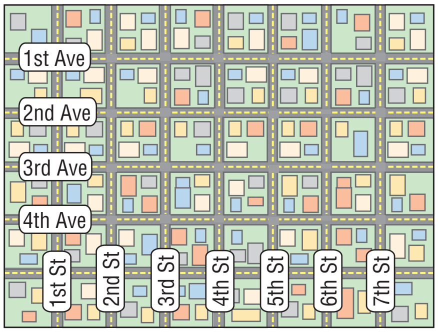
Extra Practice

See page 845.

Find the midpoint of each line segment with endpoints at the given coordinates.

10. (8, 3), (16, 7)
11. (−5, 3), (−3, −7)
12. (6, −5), (−2, −7)
13. (5, 9), (12, 18)
14. (0.45, 7), (−0.3, −0.6)
15. (4.3, −2.1), (1.9, 7.5)
16. $\left(\frac{1}{2}, -\frac{2}{3}\right), \left(\frac{1}{3}, \frac{1}{4}\right)$
17. $\left(\frac{1}{3}, \frac{3}{4}\right), \left(-\frac{1}{4}, \frac{1}{2}\right)$
18. **GEOMETRY** Triangle *MNP* has vertices *M*(3, 5), *N*(−2, 8), and *P*(7, −4). Find the coordinates of the midpoint of each side.
19. **GEOMETRY** Circle *Q* has a diameter $\overline{AB}$. If *A* is at (−3, −5) and the center is at (2, 3), find the coordinates of *B*.

20. REAL ESTATE In John's town, the numbered streets and avenues form a grid. He belongs to a gym at the corner of 12th Street and 15th Avenue, and the deli where he works is at the corner of 4th Street and 5th Avenue. He wants to rent an apartment halfway between the two. In what area should he look?



GEOGRAPHY For Exercises 21–23, use the following information.
The U.S. Geological Survey (USGS) has determined the official center of the continental United States.

21. Describe a method that might be used to approximate the geographical center of the continental United States.

22. RESEARCH Use the Internet or other reference to look up the USGS geographical center of the continental United States.

23. How does the location given by USGS compare to the result of your method?

Find the distance between each pair of points with the given coordinates.

24. $(-4, 9), (1, -3)$

25. $(1, -14), (-6, 10)$

26. $(-4, -10), (-3, -11)$

27. $(9, -2), (12, -14)$

28. $(0.23, 0.4), (0.68, -0.2)$

29. $(2.3, -1.2), (-4.5, 3.7)$

30. $\left(-3, -\frac{2}{11}\right), \left(5, \frac{9}{11}\right)$

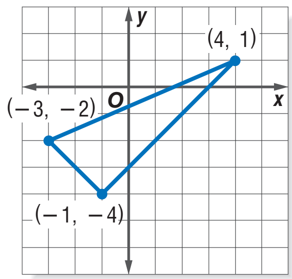
31. $\left(0, \frac{1}{5}\right), \left(\frac{3}{5}, -\frac{3}{5}\right)$

32. $\left(2\sqrt{3}, -5\right), \left(-3\sqrt{3}, 9\right)$

33. $\left(\frac{2\sqrt{3}}{3}, \frac{\sqrt{5}}{4}\right), \left(-\frac{2\sqrt{3}}{3}, \frac{\sqrt{5}}{2}\right)$

34. GEOMETRY A circle has a radius with endpoints at $(2, 5)$ and $(-1, -4)$. Find the circumference and area of the circle.

35. GEOMETRY Find the perimeter and area of the triangle shown at the right.



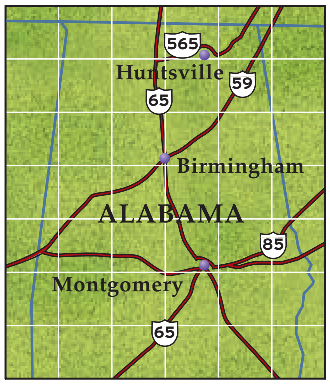
36. GEOMETRY Quadrilateral $RSTV$ has vertices $R(-4, 6)$, $S(4, 5)$, $T(6, 3)$, and $V(5, -8)$. Find the perimeter of the quadrilateral.

37. GEOMETRY Triangle CAT has vertices $C(4, 9)$, $A(8, -9)$, and $T(-6, 5)$. M is the midpoint of $\overline{TA}$. Find the length of median $\overline{CM}$. (*Hint*: A median connects a vertex of a triangle to the midpoint of the opposite side.)

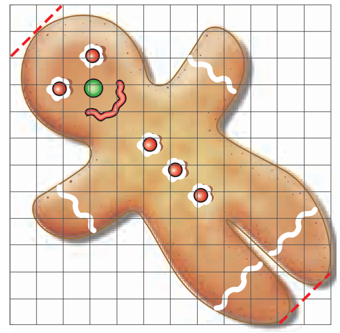
TRAVEL For Exercises 38 and 39, use the figure at the right, where a grid is superimposed on a map of a portion of the state of Alabama.

38. About how far is it from Birmingham to Montgomery if each unit on the grid represents 40 miles?

39. How long would it take a plane to fly from Huntsville to Montgomery if its average speed is 180 miles per hour?



40. **WOODWORKING** A stage crew is making the set for a children's play. They want to make some gingerbread shapes out of some leftover squares of wood with sides measuring 1 foot. They can make taller shapes by cutting them out of the wood diagonally. To the nearest inch, how tall is the gingerbread shape in the drawing at the right?



41. **CRITICAL THINKING** Verify the Midpoint Formula. (*Hint*: You must show that the formula gives the coordinates of a point on the line through the given endpoints and that the point is equidistant from the endpoints.)

42. **WRITING IN MATH** Answer the question that was posed at the beginning of the lesson.

How are the Midpoint and Distance Formulas used in emergency medicine?

Include the following in your answer:

- a few sentences explaining how to use the Distance Formula to approximate the distance between two cities on a map, and
- which city, Lincoln or Omaha, an emergency medical helicopter should be dispatched from to pick up a patient in Fremont.

Standardized Test Practice

43. What is the distance between the points $A(4, -2)$ and $B(-4, -8)$?

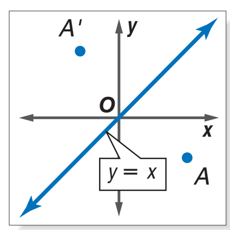
(A) 6 (B) 8 (C) 10 (D) 14

44. Point $D(5, -1)$ is the midpoint of segment $\overline{CE}$. If point C has coordinates $(3, 2)$, then what are the coordinates of point E?

(A) $(8, 1)$ (B) $(7, -4)$ (C) $(2, -3)$ (D) $\left(4, \frac{1}{2}\right)$

Extending the Lesson

For Exercises 45 and 46, use the following information.
You can use midpoints and slope to describe some transformations. Suppose point A' is the image when point A is reflected over the line with equation $y = x$.



45. Where is the midpoint of $\overline{AA'}$?

46. What is the slope of $\overleftrightarrow{AA'}$? Explain.

Maintain Your Skills

Mixed Review

Graph each function. State the domain and range. *(Lesson 7-9)*

47. $y = \sqrt{x - 2}$ 48. $y = \sqrt{x} - 1$ 49. $y = 2\sqrt{x} + 1$

50. Determine whether the functions $f(x) = x - 2$ and $g(x) = 2x$ are inverse functions. *(Lesson 7-8)*

Simplify. *(Lesson 5-9)*

51. $(2 + 4i) + (-3 + 9i)$ 52. $(4 - i) - (-2 + i)$ 53. $(1 - 2i)(2 + i)$

Getting Ready for the Next Lesson

PREREQUISITE SKILL **Write each equation in the form $y = a(x - h)^2 + k$.**
*(To review **completing the square**, see Lesson 6-4.)*

54. $y = x^2 + 6x + 9$ 55. $y = x^2 - 4x + 1$ 56. $y = 2x^2 + 20x + 50$

57. $y = 3x^2 - 6x + 5$ 58. $y = -x^2 - 4x + 6$ 59. $y = -3x^2 - 18x - 10$

Algebra Activity

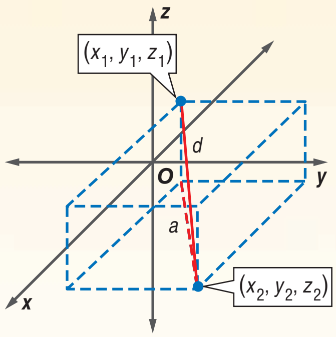
A Follow-Up of Lesson 8-1

Midpoint and Distance Formulas in Three Dimensions

You can derive a formula for distance in three-dimensional space. It may seem that the formula would involve a cube root, but it actually involves a square root, similar to the formula in two dimensions.

Suppose (x_1, y_1, z_1) and (x_2, y_2, z_2) name two points in space. Draw the rectangular box that has opposite vertices at these points. The dimensions of the box are $|x_2 - x_1|$, $|y_2 - y_1|$, and $|z_2 - z_1|$. Let a be the length of a diagonal of the bottom of the box. By the Pythagorean Theorem, $a_2 = |x_2 - x_1|^2 + |y_2 - y_1|^2$.

To find the distance d between (x_1, y_1, z_1) and (x_2, y_2, z_2), apply the Pythagorean Theorem to the right triangle whose legs are a diagonal of the bottom of the box and a vertical edge of the box.



$d^2 = a^2 + |z_2 - z_1|^2$ Pythagorean Theorem

$d^2 = |x_2 - x_1|^2 + |y_2 - y_1|^2 + |z_2 - z_1|^2$ $a_2 = |x_2 - x_1|^2 + |y_2 - y_1|^2$

$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2$ $|x_2 - x_1|^2 = (x_2 - x_1)^2$, and so on

$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$ Take the square root of each side.

The distance d between the points with coordinates (x_1, y_1, z_1) and (x_2, y_2, z_2) is given by the formula $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$.

Example 1

Find the distance between (2, 0, −3) and (4, 2, 9).

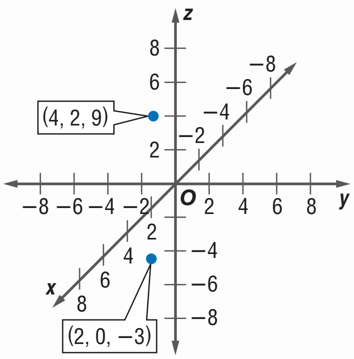
$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$ Distance Formula

$= \sqrt{(4 - 2)^2 + (2 - 0)^2 + [9 - (-3)]^2}$ $(x_1, y_1, z_1) = (2, 0, -3)$, $(x_2, y_2, z_2) = (4, 2, 9)$

$= \sqrt{2^2 + 2^2 + 12^2}$

$= \sqrt{152}$ or $2\sqrt{38}$

The distance is $2\sqrt{38}$ or about 12.33 units.



In three dimensions, the midpoint of the segment with coordinates (x_1, y_1, z_1) and (x_2, y_2, z_2) has coordinates $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$. Notice how similar this is to the Midpoint Formula in two dimensions.

(continued on the next page)

Example 2

Find the coordinates of the midpoint of the segment with endpoints (6, −5, 1) and (−2, 4, 0).

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right) = \left(\frac{6 + (-2)}{2}, \frac{-5 + 4}{2}, \frac{1 + 0}{2}\right)$$ $(x_1, y_1, z_1) = (6, -5, 1)$, $(x_2, y_2, z_2) = (-2, 4, 0)$

$$= \left(\frac{4}{2}, \frac{-1}{2}, \frac{1}{2}\right)$$ Add.

$$= \left(2, -\frac{1}{2}, \frac{1}{2}\right)$$ Simplify.

The midpoint has coordinates $\left(2, -\frac{1}{2}, \frac{1}{2}\right)$.

Exercises

Find the distance between each pair of points with the given coordinates.

1. (2, 4, 5), (1, 2, 3)

2. (−1, 6, 2), (4, −3, 0)

3. (−2, 1, 7), (−2, 6, −3)

4. (0, 7, −1), (−4, 1, 3)

Find the midpoint of each line segment with endpoints at the given coordinates.

5. (2, 6, −1), (−4, 8, 5)

6. (4, −3, 2), (−2, 7, 6)

7. (1, 3, 7), (−4, 2, −1)

8. (2.3, −1.7, 0.6), (−2.7, 3.1, 1.8)

9. The coordinates of one endpoint of a segment are (4, −2, 3), and the coordinates of the midpoint are (3, 2, 5). Find the coordinates of the other endpoint.

10. Two of the opposite vertices of a rectangular solid are at (4, 1, −1) and (2, 3, 5). Find the coordinates of the other six vertices.

11. Determine whether a triangle with vertices at (2, −4, 2), (3, 1, 5), and (6, −3, −1) is a right triangle. Explain.

The vertices of a rectangular solid are at (−2, 3, 2), (3, 3, 2), (3, 1, 2), (−2, 1, 2), (−2, 3, 6), (3, 3, 6), (3, 1, 6), and (−2, 1, 6).

12. Find the volume of the solid.

13. Find the length of a diagonal of the solid.

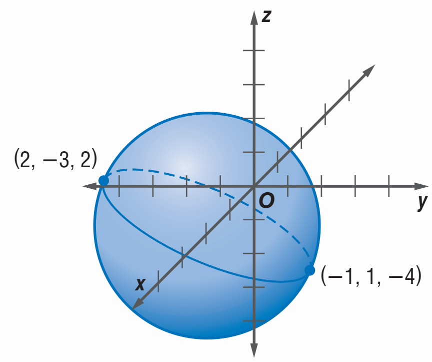
14. Show that the point with coordinates $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$ is equidistant from the points with coordinates (x_1, y_1, z_1) and (x_2, y_2, z_2).

15. Find the value of c so that the point with coordinates $(2, 3, c)$ is $3\sqrt{6}$ units from the point with coordinates (−1, 0, 5).

The endpoints of a diameter of a sphere are at (2, −3, 2) and (−1, 1, −4).

16. Find the length of a radius of the sphere.

17. Find the coordinates of the center of the sphere.



8-2 Parabolas

What You'll Learn

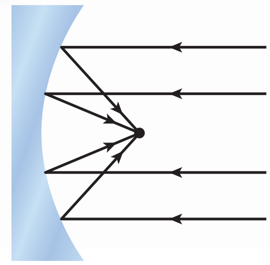
- Write equations of parabolas in standard form.
- Graph parabolas.

Vocabulary

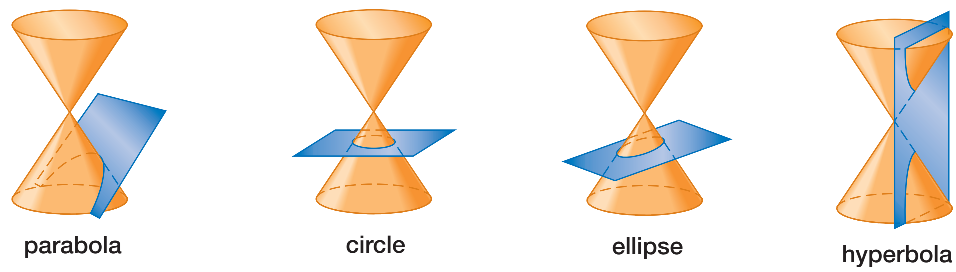
- parabola
- conic section
- focus
- directrix
- latus rectum

How are parabolas used in manufacturing?

A mirror or other reflective object in the shape of a parabola has the property that parallel incoming rays are all reflected to the same point. Or, if that point is the source of rays, the rays become parallel when they are reflected.



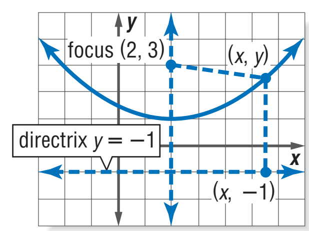
EQUATIONS OF PARABOLAS In Chapter 6, you learned that the graph of an equation of the form $y = ax^2 + bx + c$ is a **parabola**. A parabola can also be obtained by slicing a double cone on a slant as shown below on the left. Any figure that can be obtained by slicing a double cone is called a **conic section**. Other conic sections are also shown below.



Study Tip

Focus of a Parabola
The focus is the special point referred to at the beginning of the lesson.

A parabola can also be defined as the set of all points in a plane that are the same distance from a given point called the **focus** and a given line called the **directrix**. The parabola at the right has its focus at (2, 3), and the equation of its directrix is $y = -1$. You can use the Distance Formula to find an equation of this parabola.



Let (x, y) be any point on this parabola. The distance from this point to the focus must be the same as the distance from this point to the directrix. The distance from a point to a line is measured along the perpendicular from the point to the line.

distance from (x, y) to $(2, 3)$ = distance from (x, y) to $(x, -1)$

$$\sqrt{(x-2)^2 + (y-3)^2} = \sqrt{(x-x)^2 + [y-(-1)]^2}$$

$$(x-2)^2 + (y-3)^2 = 0^2 + (y+1)^2 \quad \text{Square each side.}$$

$$(x-2)^2 + y^2 - 6y + 9 = y^2 + 2y + 1 \quad \text{Square } y - 3 \text{ and } y + 1.$$

$$(x-2)^2 + 8 = 8y \quad \text{Isolate the } y\text{-terms.}$$

$$\frac{1}{8}(x-2)^2 + 1 = y \quad \text{Divide each side by 8.}$$

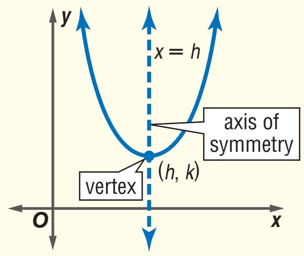
An equation of the parabola with focus at (2, 3) and directrix with equation $y = -1$ is $y = \frac{1}{8}(x - 2)^2 + 1$. The equation of the *axis of symmetry* for this parabola is $x = 2$. The axis of symmetry intersects the parabola at a point called the *vertex*. The vertex is the point where the graph turns. The vertex of this parabola is at (2, 1). Since $\frac{1}{8}$ is positive, the parabola opens upward.

Any equation of the form $y = ax^2 + bx + c$ can be written in standard form.

Key Concept — *Equation of a Parabola*

The standard form of the equation of a parabola with vertex (h, k) and axis of symmetry $x = h$ is $y = a(x - h)^2 + k$.

- If $a > 0$, k is the minimum value of the related function and the parabola opens upward.
- If $a < 0$, k is the maximum value of the related function and the parabola opens downward.



Example 1 *Analyze the Equation of a Parabola*

Write $y = 3x^2 + 24x + 50$ in standard form. Identify the vertex, axis of symmetry, and direction of opening of the parabola.

$y = 3x^2 + 24x + 50$	Original equation
$y = 3(x^2 + 8x) + 50$	Factor 3 from the x-terms.
$y = 3(x^2 + 8x + \blacksquare) + 50 - 3(\blacksquare)$	Complete the square on the right side.
$y = 3(x^2 + 8x + 16) + 50 - 3(16)$	The 16 added when you complete the square is multiplied by 3.
$y = 3(x + 4)^2 + 2$	
$y = 3[x - (-4)]^2 + 2$	$(h, k) = (-4, 2)$

The vertex of this parabola is located at $(-4, 2)$, and the equation of the axis of symmetry is $x = -4$. The parabola opens upward.

Study Tip

Look Back
To review **completing the square**, see Lesson 6-4.

GRAPH PARABOLAS You can use symmetry and translations to graph parabolas. The equation $y = a(x - h)^2 + k$ can be obtained from $y = ax^2$ by replacing x with $x - h$ and y with $y - k$. Therefore, the graph of $y = a(x - h)^2 + k$ is the graph of $y = ax^2$ translated h units to the right and k units up.

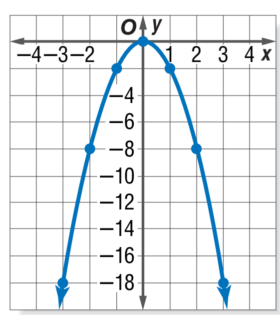
Example 2 *Graph Parabolas*

Graph each equation.

a. $y = -2x^2$

For this equation, $h = 0$ and $k = 0$. The vertex is at the origin. Since the equation of the axis of symmetry is $x = 0$, substitute some small positive integers for x and find the corresponding y-values.

x	y
1	−2
2	−8
3	−18

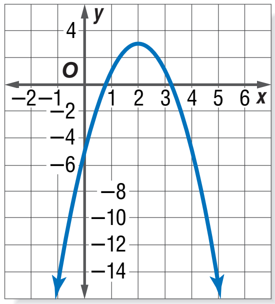


Since the graph is symmetric about the y-axis, the points $(-1, -2)$, $(-2, -8)$, and $(-3, -18)$ are also on the parabola. Use all of these points to draw the graph.

Notice that each side of the graph is the reflection of the other side about the y-axis.

b. $y = -2(x - 2)^2 + 3$

The equation is of the form $y = a(x - h)^2 + k$, where $h = 2$ and $k = 3$. The graph of this equation is the graph of $y = -2x^2$ in part **a** translated 2 units to the right and up 3 units. The vertex is now at (2, 3).



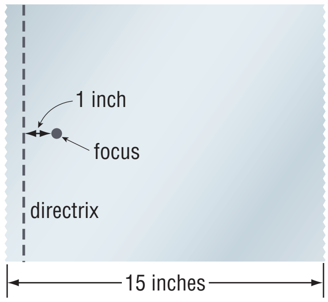
You can use paper folding to investigate the characteristics of a parabola.

Algebra Activity

Parabolas

Model

Step 1 Start with a sheet of wax paper that is about 15 inches long and 12 inches wide. Make a line that is perpendicular to the sides of the sheet by folding the sheet near one end. Open up the paper again. This line is the directrix. Mark a point about midway between the sides of the sheet so that the distance from the directrix is about 1 inch. This point is the focus.



Put the focus on top of any point on the directrix and crease the paper. Make about 20 more creases by placing the focus on top of other points on the directrix. The lines form the outline of a parabola.

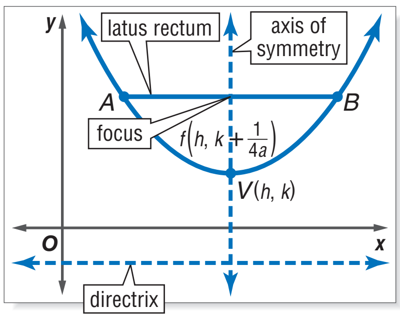
Step 2 Start with a new sheet of wax paper. Form another outline of a parabola with a focus that is about 3 inches from the directrix.

Step 3 On a new sheet of wax paper, form a third outline of a parabola with a focus that is about 5 inches from the directrix.

Analyze

Compare the shapes of the three parabolas. How does the distance between the focus and the directrix affect the shape of a parabola?

The shape of a parabola and the distance between the focus and directrix depend on the value of a in the equation. The line segment through the focus of a parabola and perpendicular to the axis of symmetry is called the **latus rectum**. The endpoints of the latus rectum lie on the parabola. In the figure at the right, the latus rectum is $\overline{AB}$. The length of the latus rectum of the parabola with equation $y = a(x - h)^2 + k$ is $\left|\frac{1}{a}\right|$ units. The endpoints of the latus rectum are $\left|\frac{1}{2a}\right|$ units from the focus.



www.algebra2.com/extra_examples

CONTENTS

Equations of parabolas with vertical axes of symmetry are of the form $y = a(x - h)^2 + k$ and are functions. Equations of parabolas with horizontal axes of symmetry are of the form $x = a(y - k)^2 + h$ and are not functions.

Concept Summary — *Information About Parabolas*

Form of Equation	$y = a(x - h)^2 + k$	$x = a(y - k)^2 + h$
Vertex	(h, k)	(h, k)
Axis of Symmetry	$x = h$	$y = k$
Focus	$\left(h, k + \frac{1}{4a}\right)$	$\left(h + \frac{1}{4a}, k\right)$
Directrix	$y = k - \frac{1}{4a}$	$x = h - \frac{1}{4a}$
Direction of Opening	upward if $a > 0$, downward if $a < 0$	right if $a > 0$, left if $a < 0$
Length of Latus Rectum	$\left\lvert \frac{1}{a} \right\rvert$ units	$\left\lvert \frac{1}{a} \right\rvert$ units

Example 3 Graph an Equation Not in Standard Form

Graph $4x - y^2 = 2y + 13$.

First, write the equation in the form $x = a(y - k)^2 + h$.

$4x - y^2 = 2y + 13$ — There is a y^2 term, so isolate the y and y^2 terms.

$4x = y^2 + 2y + 13$

$4x = (y^2 + 2y + \blacksquare) + 13 - \blacksquare$ — Complete the square.

$4x = (y^2 + 2y + 1) + 13 - 1$ — Add and subtract 1, since $\left(\frac{2}{2}\right)^2 = 1$.

$4x = (y + 1)^2 + 12$ — Write $y^2 + 2y + 1$ as a square.

$x = \frac{1}{4}(y + 1)^2 + 3$ — $(h, k) = (3, -1)$

Then use the following information to draw the graph.

vertex: $(3, -1)$

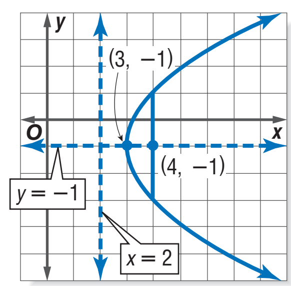
axis of symmetry: $y = -1$

focus: $\left(3 + \frac{1}{4\left(\frac{1}{4}\right)}, -1\right)$ or $(4, -1)$

directrix: $x = 3 - \frac{1}{4\left(\frac{1}{4}\right)}$ or 2

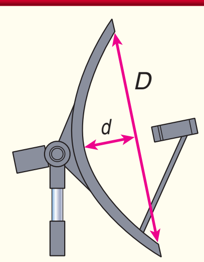
direction of opening: right, since $a > 0$

length of latus rectum: $\left\lvert \frac{1}{\frac{1}{4}} \right\rvert$ or 4 units



Remember that you can plot as many points as necessary to help you draw an accurate graph.

More About. . .



Satellite TV

The important characteristics of a satellite dish are the diameter D, depth d, and the ratio $\frac{f}{D}$, where f is the distance between the focus and the vertex. A typical dish has the values $D = 60$ cm, $d = 6.25$ cm, and $\frac{f}{D} = 0.6$.

Source: www.2000networks.com

Example 4 Write and Graph an Equation for a Parabola

SATELLITE TV **Satellite dishes have parabolic cross sections.**

a. Use the information at the left to write an equation that models a cross section of a satellite dish. Assume that the focus is at the origin and the parabola opens to the right.

First, solve for f. Since $\frac{f}{D} = 0.6$, and $D = 60$, $f = 0.6(60)$ or 36.

The focus is at $(0, 0)$, and the parabola opens to the right. So the vertex must be at $(-36, 0)$. Thus, $h = -36$ and $k = 0$. Use the x-coordinate of the focus to find a.

CONTENTS

$$-36 + \frac{1}{4a} = 0 \qquad h = -36\text{; The } x\text{-coordinate of the focus is 0.}$$

$$\frac{1}{4a} = 36 \qquad \text{Add 36 to each side.}$$

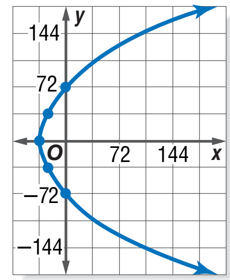
$$1 = 144a \qquad \text{Multiply each side by } 4a.$$

$$\frac{1}{144} = a \qquad \text{Divide each side by 144.}$$

An equation of the parabola is $x = \frac{1}{144}y^2 - 36$.

b. Graph the equation.

The length of the latus rectum is $\left|\frac{1}{\frac{1}{144}}\right|$ or 144 units, so the graph must pass through (0, 72) and (0, −72). According to the diameter and depth of the dish, the graph must pass through (−29.75, 30) and (−29.75, −30). Use these points and the information from part **a** to draw the graph.



Check for Understanding

Concept Check

1. **Identify** the vertex, focus, axis of symmetry, and directrix of the graph of $y = 4(x - 3)^2 - 7$.
2. **OPEN ENDED** Write an equation for a parabola that opens to the left.
3. **FIND THE ERROR** Katie is finding the standard form of the equation $y = x^2 + 6x + 4$. What mistake did she make in her work?

$y = x^2 + 6x + 4$

$y = x^2 + 6x + 9 + 4$

$y = (x + 3)^2 + 4$

Guided Practice

4. Write $y = 2x^2 - 12x + 6$ in standard form.

Identify the coordinates of the vertex and focus, the equations of the axis of symmetry and directrix, and the direction of opening of the parabola with the given equation. Then find the length of the latus rectum and graph the parabola.

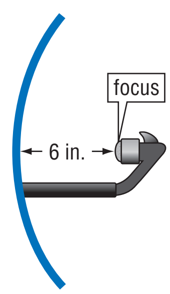
5. $y = (x - 3)^2 - 4$
6. $y = 2(x + 7)^2 + 3$
7. $y = -3x^2 - 8x - 6$
8. $x = \frac{2}{3}y^2 - 6y + 12$

Write an equation for each parabola described below. Then draw the graph.

9. focus (3, 8), directrix $y = 4$
10. vertex (5, −1), focus (3, −1)

Application

11. **COMMUNICATION** A microphone is placed at the focus of a parabolic reflector to collect sound for the television broadcast of a World Cup soccer game. Write an equation for the cross section, assuming that the focus is at the origin and the parabola opens to the right.



Practice and Apply

Homework Help

For Exercises	See Examples
12–15, 35	1
16–34	1–3
36–41	2–4
42–45	4a

Extra Practice

See page 845.

Write each equation in standard form.

12. $y = x^2 - 6x + 11$
13. $x = y^2 + 14y + 20$
14. $y = \frac{1}{2}x^2 + 12x - 8$
15. $x = 3y^2 + 5y - 9$

Identify the coordinates of the vertex and focus, the equations of the axis of symmetry and directrix, and the direction of opening of the parabola with the given equation. Then find the length of the latus rectum and graph the parabola.

16. $-6y = x^2$
17. $y^2 = 2x$
18. $3(y - 3) = (x + 6)^2$
19. $-2(y - 4) = (x - 1)^2$
20. $4(x - 2) = (y + 3)^2$
21. $(y - 8)^2 = -4(x - 4)$
22. $y = x^2 - 12x + 20$
23. $x = y^2 - 14y + 25$
24. $x = 5y^2 + 25y + 60$
25. $y = 3x^2 - 24x + 50$
26. $y = -2x^2 + 5x - 10$
27. $x = -4y^2 + 6y + 2$
28. $y = \frac{1}{2}x^2 - 3x + \frac{19}{2}$
29. $x = -\frac{1}{3}y^2 - 12y + 15$

For Exercises 30–34, use the equation $x = 3y^2 + 4y + 1$.

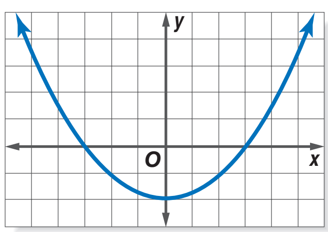
30. Draw the graph.
31. Find the x-intercept(s).
32. Find the y-intercept(s).
33. What is the equation of the axis of symmetry?
34. What are the coordinates of the vertex?

35. **MANUFACTURING** The reflective surface in a flashlight has a parabolic cross section that can be modeled by $y = \frac{1}{3}x^2$, where x and y are in centimeters. How far from the vertex should the filament of the light bulb be located?

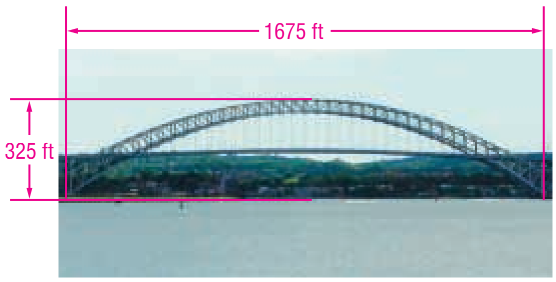
Write an equation for each parabola described below. Then draw the graph.

36. vertex (0, 1), focus (0, 5)
37. vertex (8, 6), focus (2, 6)
38. focus $(-4, -2)$, directrix $x = -8$
39. vertex (1, 7), directrix $y = 3$
40. vertex $(-7, 4)$, axis of symmetry $x = -7$, measure of latus rectum 6, $a < 0$
41. vertex (4, 3), axis of symmetry $y = 3$, measure of latus rectum 4, $a > 0$

42. Write an equation for the graph at the right.

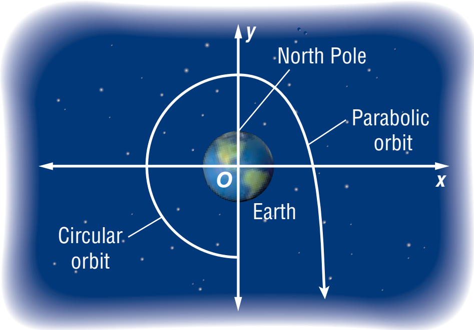


43. **BRIDGES** The Bayonne Bridge connects Staten Island, New York, to New Jersey. It has an arch in the shape of a parabola that opens downward. Write an equation of a parabola to model the arch, assuming that the origin is at the surface of the water, beneath the vertex of the arch.



44. SPORTS When a ball is thrown or kicked, the path it travels is shaped like a parabola. Suppose a football is kicked from ground level, reaches a maximum height of 25 feet, and hits the ground 100 feet from where it was kicked. Assuming that the ball was kicked at the origin, write an equation of the parabola that models the flight of the ball.

45. AEROSPACE A spacecraft is in a circular orbit 150 kilometers above Earth. Once it attains the velocity needed to escape Earth's gravity, the spacecraft will follow a parabolic path with the center of Earth as the focus. Suppose the spacecraft reaches escape velocity above the North Pole. Write an equation to model the parabolic path of the spacecraft, assuming that the center of Earth is at the origin and the radius of Earth is 6400 kilometers.



46. CRITICAL THINKING The parabola with equation $y = (x - 4)^2 + 3$ has its vertex at (4, 3) and passes through (5, 4). Find an equation of a different parabola with its vertex at (4, 3) that passes through (5, 4).

47. WRITING IN MATH Answer the question that was posed at the beginning of the lesson.

How are parabolas used in manufacturing?

Include the following in your answer:

- how you think the focus of a parabola got its name, and
- why a car headlight with a parabolic reflector is better than one with an unreflected light bulb.

Standardized Test Practice

48. Which equation has a graph that opens downward?

(A) $y = 3x^2 - 2$ (B) $y = 2 - 3x^2$ (C) $x = 3y^2 - 2$ (D) $x = 2 - 3y^2$

49. Find the vertex of the parabola with equation $y = x^2 - 10x + 8$.

(A) $(5, -17)$ (B) $(10, 8)$ (C) $(0, 8)$ (D) $(5, 8)$

Maintain Your Skills

Mixed Review

Find the distance between each pair of points with the given coordinates. *(Lesson 8-1)*

50. $(7, 3), (-5, 8)$ **51.** $(4, -1), (-2, 7)$ **52.** $(-3, 1), (0, 6)$

53. Graph $y \le \sqrt{x + 1}$. *(Lesson 7-9)*

54. HEALTH Ty's heart rate is usually 120 beats per minute when he runs. If he runs for 2 hours every day, about how many times will his heart beat during the amount of time he exercises in two weeks? Express the answer in scientific notation. *(Lesson 5-1)*

Getting Ready for the Next Lesson

PREREQUISITE SKILL **Simplify each radical expression.**
*(To review **simplifying radicals**, see Lessons 5-5 and 5-6.)*

55. $\sqrt{16}$ **56.** $\sqrt{25}$ **57.** $\sqrt{81}$ **58.** $\sqrt{144}$

59. $\sqrt{12}$ **60.** $\sqrt{18}$ **61.** $\sqrt{48}$ **62.** $\sqrt{72}$