4 Matrices

What You'll Learn

- **Lesson 4-1** Organize data in matrices.
- **Lessons 4-2, 4-3, and 4-5** Perform operations with matrices and determinants.
- **Lesson 4-4** Transform figures on a coordinate plane.
- **Lessons 4-6 and 4-8** Use matrices to solve systems of equations.
- **Lesson 4-7** Find the inverse of a matrix.

Key Vocabulary

- **matrix** (p. 154)
- **determinant** (p.182)
- **expansion by minors** (p. 183)
- **Cramer's Rule** (p. 189)
- **matrix equation** (p. 202)

Why It's Important

Data are often organized into matrices. For example, the National Federation of State High School Associations uses matrices to record student participation in sports by category for males and females. To find the total participation of both groups in each sport, you can add the two matrices. *You will learn how to add matrices in Lesson 4-2.*

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Getting Started

▶ **Prerequisite Skills** To be successful in this chapter, you'll need to master these skills and be able to apply them in problem-solving situations. Review these skills before beginning Chapter 4.

For Lesson 4-1   Solve Equations

Solve each equation. *(For review, see Lesson 1-3.)*

1. $3x = 18$ **2.** $2a - 3 = -11$ **3.** $4t - 5 = 14$

4. $\frac{1}{3}y + 5 = 9$ **5.** $3k + 5 = 2k - 8$ **6.** $5m - 6 = 7m - 8$

For Lessons 4-2 and 4-7   Additive and Multiplicative Inverses

Name the additive inverse and the multiplicative inverse for each number. *(For review, see Lesson 1-2.)*

7. 3 **8.** -11 **9.** 8 **10.** -0.5

11. 1.25 **12.** $\frac{5}{9}$ **13.** $-\frac{8}{3}$ **14.** $-1\frac{1}{5}$

For Lesson 4-4   Graph Ordered Pairs

Graph each set of ordered pairs on a coordinate plane. *(For review, see Lesson 2-1.)*

15. $\{(0, 0), (1, 3), (-2, 4)\}$ **16.** $\{(-1, 5), (2, -3), (4, 0)\}$

17. $\{(-3, -3), (-1, 2), (1, -3), (3, -6)\}$ **18.** $\{(-2, 5), (1, 3), (4, -2), (4, 7)\}$

For Lessons 4-6 and 4-8   Solve Systems of Equations

Solve each system of equations by using either substitution or elimination. *(For review, see Lesson 3-2.)*

19. $x = y + 5$
$3x + y = 19$ **20.** $3x - 2y = 1$
$4x + 2y = 20$ **21.** $5x + 3y = 25$
$4x + 7y = -3$

22. $y = x - 7$
$2x - 8y = 2$ **23.** $5x - 3y = 16$
$x - 3y = 8$ **24.** $9x + 4y = 17$
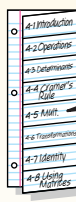$3x - 2y = 29$

FOLDABLES™
Study Organizer

Make this Foldable to record information about matrices. Begin with one sheet of notebook paper.

Step 1   Fold and Cut

Fold lengthwise to the holes. Cut eight tabs in the top sheet.

Step 2   Label

Label each tab with a lesson number and title.

4-1 Introduction
4-2 Operations
4-3 Determinants
4-4 Cramer's Rule
4-5 Mult.
4-6 Transformations
4-7 Identity
4-8 Using Matrices

Reading and Writing As you read and study the chapter, write notes and examples for each topic under the tabs.

Introduction to Matrices

What You'll Learn

- Organize data in matrices.
- Solve equations involving matrices.

Vocabulary

- matrix
- element
- dimension
- row matrix
- column matrix
- square matrix
- zero matrix
- equal matrices

How are matrices used to make decisions?

Sabrina wants to buy a sports-utility vehicle (SUV). There are many types of SUVs in many prices and styles. So, Sabrina makes a list of the qualities for different models and organizes the information in a matrix.

	Base Price	Horse-power	Towing Capacity (lb)	Cargo Space (ft³)	Fuel Economy (mpg)
Large SUV	$32,450	285	12,000	46	17
Standard SUV	$29,115	275	8700	16	17.5
Mid-Size SUV	$27,975	190	5700	34	20
Compact SUV	$18,180	127	3000	15	26.5

Source: *Car and Driver Buyer's Guide*

When the information is organized in a matrix, it is easy to compare the features of each vehicle.

Study Tip

Reading Math
The plural of *matrix* is *matrices*.

ORGANIZE DATA A **matrix** is a rectangular array of variables or constants in horizontal rows and vertical columns, usually enclosed in brackets.

Example 1 Organize Data in a Matrix

Sharon wants to install cable television in her new apartment. There are two cable companies in the area whose prices are listed below. Use a matrix to organize the information. When is each company's service less expensive?

Metro Cable	
Basic Service (26 channels)	$11.95
Standard Service (53 channels)	$30.75
Premium Channels (in addition to Standard Service)	
• One Premium	$10.00
• Two Premiums	$19.00
• Three Premiums	$25.00

Cable City	
Basic Service (26 channels)	$9.95
Standard Service (53 channels)	$31.95
Premium Channels (in addition to Standard Service)	
• One Premium	$8.95
• Two Premiums	$16.95
• Three Premiums	$22.95

Organize the costs into labeled columns and rows.

	Basic	Standard	Standard Plus One Premium	Standard Plus Two Premiums	Standard Plus Three Premiums
Metro Cable	11.95	30.75	40.75	49.75	55.75
Cable City	9.95	31.95	40.90	48.90	54.90

Metro Cable has the best price for standard service and standard plus one premium channel. Cable City has the best price for the other categories.

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In a matrix, numbers or data are organized so that each position in the matrix has a purpose. Each value in the matrix is called an **element**. A matrix is usually named using an uppercase letter.

$$A = \begin{bmatrix} 2 & 6 & 1 \\ 7 & 1 & 5 \\ 9 & 3 & 0 \\ 12 & 15 & 26 \end{bmatrix} \text{ 4 rows}$$

3 columns

The element 15 is in row 4, column 2.

A matrix can be described by its **dimensions**. A matrix with m rows and n columns is an $m \times n$ matrix (read "m by n"). Matrix A above is a 4×3 matrix since it has 4 rows and 3 columns.

Example 2 Dimensions of a Matrix

State the dimensions of matrix B if $B = \begin{bmatrix} 1 & -3 \\ -5 & 18 \\ 0 & -2 \end{bmatrix}$.

$$B = \begin{bmatrix} 1 & -3 \\ -5 & 18 \\ 0 & -2 \end{bmatrix} \text{ 3 rows}$$

2 columns

Since matrix B has 3 rows and 2 columns, the dimensions of matrix B are 3×2.

Certain matrices have special names. A matrix that has only one row is called a **row matrix**, while a matrix that has only one column is called a **column matrix**. A matrix that has the same number of rows and columns is called a **square matrix**. Another special type of matrix is the **zero matrix**, in which every element is 0. The zero matrix can have any dimension.

EQUATIONS INVOLVING MATRICES   Two matrices are considered **equal matrices** if they have the same dimensions and if each element of one matrix is equal to the corresponding element of the other matrix.

$$\begin{bmatrix} 6 & 3 \\ 0 & 9 \\ 1 & 3 \end{bmatrix} \neq \begin{bmatrix} 6 & 0 & 1 \\ 3 & 9 & 3 \end{bmatrix}$$
The matrices have different dimensions. They are not equal.

$$\begin{bmatrix} 1 & 2 \\ 8 & 5 \end{bmatrix} \neq \begin{bmatrix} 1 & 8 \\ 2 & 5 \end{bmatrix}$$
Corresponding elements are not equal. The matrices are not equal.

$$\begin{bmatrix} 5 & 6 & 0 \\ 0 & 7 & 2 \\ 3 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 5 & 6 & 0 \\ 0 & 7 & 2 \\ 3 & 1 & 4 \end{bmatrix}$$
The matrices have the same dimensions and the corresponding elements are equal. The matrices are equal.

The definition of equal matrices can be used to find values when elements of equal matrices are algebraic expressions.

Example 3 Solve an Equation Involving Matrices

Solve $\begin{bmatrix} y \\ 3x \end{bmatrix} = \begin{bmatrix} 6 - 2x \\ 31 + 4y \end{bmatrix}$ for x and y.

Since the matrices are equal, the corresponding elements are equal. When you write the sentences to show this equality, two linear equations are formed.

$y = 6 - 2x$

$3x = 31 + 4y$

www.algebra2.com/extra_examples

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Look Back
To review **solving
systems of equations by
using substitution**, see
Lesson 3-2.

This system can be solved using substitution.

$3x = 31 + 4y$ Second equation

$3x = 31 + 4(6 - 2x)$ Substitute $6 - 2x$ for y.

$3x = 31 + 24 - 8x$ Distributive Property

$11x = 55$ Add $8x$ to each side.

$x = 5$ Divide each side by 11.

To find the value for y, substitute 5 for x in either equation.

$y = 6 - 2x$ First equation

$y = 6 - 2(5)$ Substitute 5 for x.

$y = -4$ Simplify.

The solution is $(5, -4)$.

Check for Understanding

Concept Check
1. **Describe** the conditions that must be met in order for two matrices to be considered equal.

2. **OPEN ENDED** Give examples of a row matrix, a column matrix, a square matrix, and a zero matrix. State the dimensions of each matrix.

3. **Explain** what is meant by corresponding elements.

Guided Practice
State the dimensions of each matrix.

4. $[3 \quad 4 \quad 5 \quad 6 \quad 7]$

5. $\begin{bmatrix} 10 & -6 & 18 & 0 \\ -7 & 5 & 2 & 4 \\ 3 & 11 & 9 & 7 \end{bmatrix}$

Solve each equation.

6. $\begin{bmatrix} x + 4 \\ 2y \end{bmatrix} = \begin{bmatrix} 9 \\ 12 \end{bmatrix}$

7. $[9 \quad 13] = [x + 2y \quad 4x + 1]$

Application
WEATHER For Exercises 8 and 9, use the table that shows a five-day forecast indicating high (H) and low (L) temperatures.

8. Organize the temperatures in a matrix.

9. What are the dimensions of the matrix?

Fri	Sat	Sun	Mon	Tue
☀	☀	☀	⛅	⛅
H 88	H 88	H 90	H 86	H 85
L 54	L 54	L 56	L 53	L 52

Practice and Apply

State the dimensions of each matrix.

10. $\begin{bmatrix} 6 & -1 & 5 \\ -2 & 3 & -4 \end{bmatrix}$

11. $\begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$

12. $\begin{bmatrix} 0 & 0 & 8 \\ 6 & 2 & 4 \\ 1 & 3 & 6 \\ 5 & 9 & 2 \end{bmatrix}$

13. $\begin{bmatrix} -3 & 17 & -22 \\ 9 & 31 & 16 \\ 20 & -15 & 4 \end{bmatrix}$

14. $\begin{bmatrix} 17 & -2 & 8 & -9 & 6 \\ 5 & 11 & 20 & -1 & 4 \end{bmatrix}$

15. $\begin{bmatrix} 16 & 8 \\ 10 & 5 \\ 0 & 0 \end{bmatrix}$

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Solve each equation.

16. $[2x \quad 3 \quad 3z] = [5 \quad 3y \quad 9]$

17. $[4x \quad 3y] = [12 \quad -1]$

18. $\begin{bmatrix} 4x \\ 5 \end{bmatrix} = \begin{bmatrix} 15 + x \\ 2y - 1 \end{bmatrix}$

19. $\begin{bmatrix} 4x - 3 & 3y \\ 7 & 13 \end{bmatrix} = \begin{bmatrix} 9 & -15 \\ 7 & 2z + 1 \end{bmatrix}$

20. $\begin{bmatrix} x + 3y \\ 3x + y \end{bmatrix} = \begin{bmatrix} -13 \\ 1 \end{bmatrix}$

21. $\begin{bmatrix} 2x + y \\ x - 3y \end{bmatrix} = \begin{bmatrix} 5 \\ 13 \end{bmatrix}$

22. $\begin{bmatrix} 2x \\ 2x + 3y \end{bmatrix} = \begin{bmatrix} y \\ 12 \end{bmatrix}$

23. $\begin{bmatrix} 4x \\ y - 1 \end{bmatrix} = \begin{bmatrix} 11 + 3y \\ x \end{bmatrix}$

24. $\begin{bmatrix} x^2 + 1 & 5 - y \\ x + y & y - 4 \end{bmatrix} = \begin{bmatrix} 5 & x \\ 5 & 3 \end{bmatrix}$

25. $\begin{bmatrix} 3x - 5 & x + y \\ 12 & 9z \end{bmatrix} = \begin{bmatrix} 10 & 8 \\ 12 & 3x + y \end{bmatrix}$

MOVIES For Exercises 26 and 27, use the advertisement shown at the right.

26. Write a matrix for the prices of movie tickets for adults, children, and seniors.

27. What are the dimensions of the matrix?

NOW PLAYING
Ticket Information

Evening Shows
Adult.......$7.50
Child.......$4.50
Senior.....$5.50

Matinee Shows
Adult.......$5.50
Child.......$4.50
Senior.....$5.50

Twilight Shows
All tickets....$3.75

DINING OUT For Exercises 28 and 29, use the following information.
A newspaper rated several restaurants by cost, level of service, atmosphere, and location using a scale of ★ being low and ★★★★ being high.

Catalina Grill: cost ★★, service ★, atmosphere ★, location ★

Oyster Club: cost ★★★, service ★★, atmosphere ★, location ★★

Casa di Pasta: cost ★★★★, service ★★★, atmosphere ★★★, location ★★★

Mason's Steakhouse: cost ★★, service ★★★★, atmosphere ★★★★, location ★★★

28. Write a 4 × 4 matrix to organize this information.

29. Which restaurant would you select based on this information and why?

HOTELS For Exercises 30 and 31, use the costs for an overnight stay at a hotel that are given below.

Single Room: $60 weekday; $79 weekend

Double Room: $70 weekday; $89 weekend

Suite: $75 weekday; $95 weekend

30. Write a 3 × 2 matrix that represents the cost of each room.

31. Write a 2 × 3 matrix that represents the cost of each room.

CRITICAL THINKING For Exercises 32 and 33, use the matrix at the right.

32. Study the pattern of numbers. Complete the matrix for column 6 and row 7.

33. In which row and column will 100 occur?

$$\begin{bmatrix} 1 & 3 & 6 & 10 & 15 & \dots \\ 2 & 5 & 9 & 14 & 20 & \dots \\ 4 & 8 & 13 & 19 & 26 & \dots \\ 7 & 12 & 18 & 25 & 33 & \dots \\ 11 & 17 & 24 & 32 & 41 & \dots \\ 16 & 23 & 31 & 40 & 50 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

www.algebra2.com/self_check_quiz

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34. WRITING IN MATH   Answer the question that was posed at the beginning of the lesson.

How are matrices used to make decisions?

Include the following in your answer:
- the circumstances under which each vehicle best fits a person's needs, and
- an example of how matrices are used in newspapers or magazines.

35. In matrix $A = \begin{bmatrix} 1 & 5 & -2 \\ -4 & 0 & 6 \\ 3 & 7 & 8 \end{bmatrix}$, element 3 is in which row and column?

Ⓐ row 1, column 3   Ⓑ row 3, column 1
Ⓒ row 1, column 1   Ⓓ row 3, column 3

36. What is the value of y if $\begin{bmatrix} 3x \\ y+5 \end{bmatrix} = \begin{bmatrix} 9+y \\ x \end{bmatrix}$?

Ⓐ 2   Ⓑ 4   Ⓒ -3 Ⓓ -1

Maintain Your Skills

Mixed Review   Solve each system of equations.   *(Lesson 3-5)*

37. $3x - 3y = 6$
$-6y = -30$
$5z - 2x = 6$

38. $3a + 2b = 27$
$5a - 7b + c = 5$
$-2a + 10b + 5c = -29$

39. $3r - 15s + 4t = -57$
$9r + 45s - t = 26$
$-6r + 10s + 3t = -19$

Graph each system of inequalities. Name the coordinates of the vertices of the feasible region. Find the maximum and minimum values of the given function for this region.   *(Lesson 3-4)*

40. $y \geq 3$
$y \leq x + 2$
$y \leq -2x + 15$
$f(x, y) = 2x + 3y$

41. $y \geq \frac{1}{3}x$
$y \geq -5x + 16$
$y \leq -x + 10$
$f(x, y) = 5x - y$

42. $y \geq \frac{1}{2}x$
$y \geq -x + 3$
$y \leq -\frac{3}{2}x + 12$
$f(x, y) = 3y - x$

BUSINESS   For Exercises 43–45, use the following information.
The parking garage at Burrough's Department Store charges $1.50 for each hour or fraction of an hour for parking.   *(Lesson 2-6)*

43. Graph the function.

44. What type of function represents this situation?

45. Jada went shopping at Burrough's Department Store yesterday. She left her car in the parking garage for two hours and twenty-seven minutes. How much did Jada pay for parking?

Find each value if $f(x) = x^2 - 3x + 2$.   *(Lesson 2-1)*

46. $f(3)$ **47.** $f(0)$ **48.** $f(2)$ **49.** $f(-3)$

Getting Ready for the Next Lesson

PREREQUISITE SKILL   Find the value of each expression.
(To review evaluating expressions, see Lesson 1-2.)

50. $8 + (-5)$ **51.** $-2 - 8$ **52.** $3.5 + 2.7$ **53.** $6(-3)$

54. $\frac{1}{2}(34)$ **55.** $6(4) + 3(-9)$ **56.** $-5(3 - 18)$ **57.** $14\left(\frac{1}{4}\right) - 12\left(\frac{1}{6}\right)$

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Organizing Data

You can use a computer spreadsheet to organize and display data. Then you can use the data to create graphs or perform calculations.

Example

Enter the data on the Atlantic Coast Conference Men's Basketball scoring into a spreadsheet.

Atlantic Coast Conference 2000–2001 Men's Basketball			
Team	Free Throws	2-Point Field Goals	3-Point Field Goals
Clemson	456	549	248
Duke	697	810	407
Florida State	453	594	148
Georgia Tech	457	516	260
Maryland	622	915	205
North Carolina	532	756	189
North Carolina State	507	562	170
Virginia	556	648	204
Wake Forest	443	661	177

Source: Atlantic Coast Conference

Use Column A for the team names, Column B for the numbers of free throws, Column C for the numbers of 2-point field goals, and Column D for the numbers of 3-point field goals.

Each **row** contains data for a different team. Row 2 represents Duke.

Each **cell** of the spreadsheet contains one piece of data. Cell 8D contains the value 204, representing the number of 3-point field goals made by Virginia.

Model and Analyze

1. Enter the data about sports-utility vehicles on page 154 into a spreadsheet.
2. Compare and contrast how data are organized in a spreadsheet and how they are organized in a matrix.

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4-2 Operations with Matrices

What You'll Learn

- Add and subtract matrices.
- Multiply by a matrix scalar.

How can matrices be used to calculate daily dietary needs?

In her job as a hospital dietician, Celeste designs weekly menus for her patients and tracks various nutrients for each daily diet. The table shows the Calories, protein, and fat in a patient's meals over a three-day period.

Day	Breakfast			Lunch			Dinner		
	Calories	Protein (g)	Fat (g)	Calories	Protein (g)	Fat (g)	Calories	Protein (g)	Fat (g)
1	566	18	7	785	22	19	1257	40	26
2	482	12	17	622	23	20	987	32	45
3	530	10	11	710	26	12	1380	29	38

These data can be organized in three matrices representing breakfast, lunch, and dinner. The daily totals can then be found by adding the three matrices.

ADD AND SUBTRACT MATRICES Matrices can be added if and only if they have the same dimensions.

Key Concept — Addition of Matrices

- **Words** If A and B are two $m \times n$ matrices, then $A + B$ is an $m \times n$ matrix in which each element is the sum of the corresponding elements of A and B.

- **Symbols**
$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} + \begin{bmatrix} j & k & l \\ m & n & o \\ p & q & r \end{bmatrix} = \begin{bmatrix} a+j & b+k & c+l \\ d+m & e+n & f+o \\ g+p & h+q & i+r \end{bmatrix}$$

Example 1 Add Matrices

a. Find $A + B$ if $A = \begin{bmatrix} 4 & -6 \\ 2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} -3 & 7 \\ 5 & -9 \end{bmatrix}$.

$A + B = \begin{bmatrix} 4 & -6 \\ 2 & 3 \end{bmatrix} + \begin{bmatrix} -3 & 7 \\ 5 & -9 \end{bmatrix}$ Definition of matrix addition

$\quad = \begin{bmatrix} 4 + (-3) & -6 + 7 \\ 2 + 5 & 3 + (-9) \end{bmatrix}$ Add corresponding elements.

$\quad = \begin{bmatrix} 1 & 1 \\ 7 & -6 \end{bmatrix}$ Simplify.

b. Find $A + B$ if $A = \begin{bmatrix} 3 & -7 & 4 \\ 12 & 5 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 9 \\ 4 & -6 \end{bmatrix}$.

Since the dimensions of A are 2×3 and the dimensions of B are 2×2, you cannot add these matrices.

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You can subtract matrices in a similar manner.

Key Concept — Subtraction of Matrices

- **Words** If A and B are two $m \times n$ matrices, then $A - B$ is an $m \times n$ matrix in which each element is the difference of the corresponding elements of A and B.

- **Symbols**
$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} - \begin{bmatrix} j & k & l \\ m & n & o \\ p & q & r \end{bmatrix} = \begin{bmatrix} a-j & b-k & c-l \\ d-m & e-n & f-o \\ g-p & h-q & i-r \end{bmatrix}$$

Example 2 Subtract Matrices

Find $A - B$ if $A = \begin{bmatrix} 9 & 2 \\ -4 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 6 \\ 8 & -2 \end{bmatrix}$.

$A - B = \begin{bmatrix} 9 & 2 \\ -4 & 7 \end{bmatrix} - \begin{bmatrix} 3 & 6 \\ 8 & -2 \end{bmatrix}$ Definition of matrix subtraction

$\qquad = \begin{bmatrix} 9-3 & 2-6 \\ -4-8 & 7-(-2) \end{bmatrix}$ Subtract corresponding elements.

$\qquad = \begin{bmatrix} 6 & -4 \\ -12 & 9 \end{bmatrix}$ Simplify.

More About. . .

Animals

The rarest animal in the world today is a giant tortoise that lives in the Galapagos Islands. "Lonesome George" is the only remaining representative of his species (*Geochelone elephantopus abingdoni*). With virtually no hope of discovering another specimen, this species is now effectively extinct.

Source: www.ecoworld.com

Example 3 Use Matrices to Model Real-World Data

ANIMALS The table below shows the number of endangered and threatened species in the United States and in the world. How many more endangered and threatened species are there on the world list than on the U.S. list?

Endangered and Threatened Species				
Type of Animal	United States		World	
	Endangered	Threatened	Endangered	Threatened
Mammals	61	8	309	24
Birds	74	15	252	21
Reptiles	14	22	79	36
Amphibians	9	8	17	9
Fish	69	42	80	42

Source: Fish and Wildlife Service, U.S. Department of Interior

The data in the table can be organized in two matrices. Find the difference of the matrix that represents species in the world and the matrix that represents species in the U.S.

$$\overset{\text{World}}{\begin{bmatrix} 309 & 24 \\ 252 & 21 \\ 79 & 36 \\ 17 & 9 \\ 80 & 42 \end{bmatrix}} - \overset{\text{U.S.}}{\begin{bmatrix} 61 & 8 \\ 74 & 15 \\ 14 & 22 \\ 9 & 8 \\ 69 & 42 \end{bmatrix}} = \begin{bmatrix} 309-61 & 24-8 \\ 252-74 & 21-15 \\ 79-14 & 36-22 \\ 17-9 & 9-8 \\ 80-69 & 42-42 \end{bmatrix}$$ Subtract corresponding elements.

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$$= \begin{bmatrix} 248 & 16 \\ 178 & 6 \\ 65 & 14 \\ 8 & 1 \\ 11 & 0 \end{bmatrix} \begin{matrix} \text{mammals} \\ \text{birds} \\ \text{reptiles} \\ \text{amphibians} \\ \text{fish} \end{matrix}$$

$$\begin{matrix} \text{Endangered} & \text{Threatened} \end{matrix}$$

The first column represents the difference in the number of endangered species on the world and U.S. lists. There are 248 mammals, 178 birds, 65 reptiles, 8 amphibians, and 11 fish species in this category.

The second column represents the difference in the number of threatened species on the world and U.S. lists. There are 16 mammals, 6 birds, 14 reptiles, 1 amphibian, and no fish in this category.

SCALAR MULTIPLICATION You can multiply any matrix by a constant called a **scalar**. This operation is called **scalar multiplication**.

Key Concept *Scalar Multiplication*

- **Words** The product of a scalar k and an $m \times n$ matrix is an $m \times n$ matrix in which each element equals k times the corresponding elements of the original matrix.

- **Symbols** $k\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} = \begin{bmatrix} ka & kb & kc \\ kd & ke & kf \end{bmatrix}$

Example 4 *Multiply a Matrix by a Scalar*

If $A = \begin{bmatrix} 2 & 8 & -3 \\ 5 & -9 & 2 \end{bmatrix}$, find $3A$.

$$3A = 3\begin{bmatrix} 2 & 8 & -3 \\ 5 & -9 & 2 \end{bmatrix} \qquad \text{Substitution.}$$

$$= \begin{bmatrix} 3(2) & 3(8) & 3(-3) \\ 3(5) & 3(-9) & 3(2) \end{bmatrix} \qquad \text{Multiply each element by 3.}$$

$$= \begin{bmatrix} 6 & 24 & -9 \\ 15 & -27 & 6 \end{bmatrix} \qquad \text{Simplify.}$$

Many properties of real numbers also hold true for matrices.

The matrix $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is called a *zero matrix*. It is the *additive identity* matrix for any 2×2 matrix. How is this similar to the additive identity for real numbers?

Study Tip

Additive Identity

The matrix $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is called a *zero matrix*. It is the *additive identity* matrix for any 2×2 matrix. How is this similar to the additive identity for real numbers?

Concept Summary *Properties of Matrix Operations*

For any matrices A, B, and C with the same dimensions and any scalar c, the following properties are true.

Commutative Property of Addition	$A + B = B + A$
Associative Property of Addition	$(A + B) + C = A + (B + C)$
Distributive Property	$c(A + B) = cA + cB$

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Example 5 Combination of Matrix Operations

If $A = \begin{bmatrix} 7 & 3 \\ -4 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 9 & 6 \\ 3 & 10 \end{bmatrix}$, find $5A - 2B$.

Perform the scalar multiplication first. Then subtract the matrices.

$5A - 2B = 5\begin{bmatrix} 7 & 3 \\ -4 & -1 \end{bmatrix} - 2\begin{bmatrix} 9 & 6 \\ 3 & 10 \end{bmatrix}$ Substitution

$= \begin{bmatrix} 5(7) & 5(3) \\ 5(-4) & 5(-1) \end{bmatrix} - \begin{bmatrix} 2(9) & 2(6) \\ 2(3) & 2(10) \end{bmatrix}$ Multiply each element in the first matrix by 5 and multiply each element in the second matrix by 2.

$= \begin{bmatrix} 35 & 15 \\ -20 & -5 \end{bmatrix} - \begin{bmatrix} 18 & 12 \\ 6 & 20 \end{bmatrix}$ Simplify.

$= \begin{bmatrix} 35 - 18 & 15 - 12 \\ -20 - 6 & -5 - 20 \end{bmatrix}$ Subtract corresponding elements.

$= \begin{bmatrix} 17 & 3 \\ -26 & -25 \end{bmatrix}$ Simplify.

Study Tip

Matrix Operations
The order of operations for matrices is similar to that of real numbers. Perform scalar multiplication before matrix addition and subtraction.

Graphing Calculator Investigation

Matrix Operations

Most graphing calculators can perform operations with matrices. On the TI-83 Plus, 2nd [MATRX] accesses the matrix menu. Choose EDIT to define a matrix. Enter the dimensions of the matrix A using the ▶ key. Then enter each element by pressing ENTER after each entry. To display and use the matrix in calculations, choose the matrix under NAMES from the [MATRX] menu.

Think and Discuss

1. Enter $A = \begin{bmatrix} 3 & -2 \\ 5 & 4 \end{bmatrix}$ with a graphing calculator. Does the calculator enter elements row by row or column by column?

2. Notice that there are two numbers in the bottom left corner of the screen. What do these numbers represent?

3. Clear the screen. Find the matrix 18A.

4. Enter $B = \begin{bmatrix} 1 & 9 & -3 \\ 8 & 6 & -5 \end{bmatrix}$. Find $A + B$. What is the result and why?

Check for Understanding

Concept Check

1. **Describe** the conditions under which matrices can be added or subtracted.

2. **OPEN ENDED** Give an example of two matrices whose sum is a zero matrix.

3. **Write** a matrix that, when added to a 3×2 matrix, increases each element in the matrix by 4.

Guided Practice

Perform the indicated matrix operations. If the matrix does not exist, write *impossible*.

4. $[5 \quad 8 \quad -4] + [12 \quad 5]$

5. $\begin{bmatrix} 3 & 7 \\ -2 & 1 \end{bmatrix} - \begin{bmatrix} 2 & -3 \\ 5 & -4 \end{bmatrix}$

6. $3\begin{bmatrix} 6 & -1 & 5 & 2 \\ 7 & 3 & -2 & 8 \end{bmatrix}$

7. $4\begin{bmatrix} 2 & 7 \\ -3 & 6 \end{bmatrix} + 5\begin{bmatrix} -6 & -4 \\ 3 & 0 \end{bmatrix}$

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Use matrices A, B, and C to find the following.

$$A = \begin{bmatrix} 2 & 3 \\ 5 & 6 \end{bmatrix} \qquad B = \begin{bmatrix} -1 & 7 \\ 0 & -4 \end{bmatrix} \qquad C = \begin{bmatrix} 9 & -4 \\ -6 & 5 \end{bmatrix}$$

8. $A + B + C$ **9.** $3B - 2C$ **10.** $4A + 2B - C$

Application **SPORTS** For Exercises 11–13, use the table below that shows high school participation in various sports.

Sport	Males		Females	
	Schools	Participants	Schools	Participants
Basketball	16,763	549,499	16,439	456,873
Track and Field	14,620	477,960	14,545	405,163
Baseball/Softball	14,486	455,305	12,679	340,480
Soccer	9041	321,416	7931	257,586
Swimming and Diving	5234	83,411	5450	133,235

Source: National Federation of State High School Associations

11. Write two matrices that represent these data for males and females.

12. Find the total number of students that participate in each individual sport expressed as a matrix.

13. Could you add the two matrices to find the total number of schools that offer a particular sport? Why or why not?

Practice and Apply

Perform the indicated matrix operations. If the matrix does not exist, write impossible.

14. $\begin{bmatrix} 4 \\ 1 \\ -3 \end{bmatrix} + \begin{bmatrix} 6 \\ -5 \\ 8 \end{bmatrix}$

15. $\begin{bmatrix} -5 & 7 \\ 6 & 8 \end{bmatrix} - \begin{bmatrix} 4 & 0 & -2 \\ 9 & 0 & 1 \end{bmatrix}$

16. $\begin{bmatrix} 12 & 0 & 8 \\ 9 & 15 & -11 \end{bmatrix} - \begin{bmatrix} -3 & 0 & 4 \\ 9 & 2 & -6 \end{bmatrix}$

17. $-2 \begin{bmatrix} 2 & -4 & 1 \\ -3 & 5 & 8 \\ 7 & 6 & -2 \end{bmatrix}$

18. $5[0 \quad -1 \quad 7 \quad 2] + 3[5 \quad -8 \quad 10 \quad -4]$

19. $5 \begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix} + 6 \begin{bmatrix} -4 \\ 3 \\ 5 \end{bmatrix} - 2 \begin{bmatrix} -3 \\ 8 \\ -4 \end{bmatrix}$

20. $\begin{bmatrix} 1.35 & 5.80 \\ 1.24 & 14.32 \\ 6.10 & 35.26 \end{bmatrix} + \begin{bmatrix} 0.45 & 3.28 \\ 1.94 & 16.72 \\ 4.31 & 21.30 \end{bmatrix}$

21. $8 \begin{bmatrix} 0.25 & 0.5 \\ 0.75 & 1.5 \end{bmatrix} - 2 \begin{bmatrix} 0.25 & 0.5 \\ 0.75 & 1.5 \end{bmatrix}$

22. $\dfrac{1}{2} \begin{bmatrix} 4 & 6 \\ 3 & 0 \end{bmatrix} - \dfrac{2}{3} \begin{bmatrix} 9 & 27 \\ 0 & 3 \end{bmatrix}$

23. $5 \begin{bmatrix} \frac{1}{2} & 0 & 1 \\ 2 & \frac{1}{3} & -1 \end{bmatrix} + 4 \begin{bmatrix} -2 & \frac{3}{4} & 1 \\ \frac{1}{6} & 0 & \frac{5}{8} \end{bmatrix}$

Use matrices A, B, C, and D to find the following.

$$A = \begin{bmatrix} 5 & 7 \\ -1 & 6 \\ 3 & -9 \end{bmatrix} \qquad B = \begin{bmatrix} 8 & 3 \\ 5 & 1 \\ 4 & 4 \end{bmatrix} \qquad C = \begin{bmatrix} 0 & 4 \\ -2 & 5 \\ 7 & -1 \end{bmatrix} \qquad D = \begin{bmatrix} 6 & 2 \\ 9 & 0 \\ -3 & 0 \end{bmatrix}$$

24. $A + B$ **25.** $D - B$ **26.** $4C$

27. $6B - 2A$ **28.** $3C - 4A + B$ **29.** $C + \dfrac{1}{3}D$

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BUSINESS For Exercises 30–32, use the following information.
The Cookie Cutter Bakery records each type of cookie sold at three of their branch stores. Two days of sales are shown in the spreadsheets below.

30. Write a matrix for each day's sales.

	A	B	C	D	E
1	**Friday**	chocolate chip	peanut butter	sugar	cut-out
2	**Store 1**	120	97	64	75
3	**Store 2**	80	59	36	60
4	**Store 3**	72	84	29	48

31. Find the sum of the two days' sales expressed as a matrix.

32. Find the difference in cookie sales from Friday to Saturday expressed as a matrix.

	A	B	C	D	E
1	**Saturday**	chocolate chip	peanut butter	sugar	cut-out
2	**Store 1**	112	87	56	74
3	**Store 2**	84	65	39	70
4	**Store 3**	88	98	43	60

WEATHER For Exercises 33–35, use the table that shows the total number of deaths due to severe weather.

Year	Lightning	Tornadoes	Floods	Hurricanes
1996	52	25	131	37
1997	42	67	118	1
1998	44	130	136	9
1999	46	94	68	19
2000	51	29	37	0

Source: National Oceanic & Atmospheric Administration

33. Find the total number of deaths due to severe weather for each year expressed as a column matrix.

34. Write a matrix that represents how many more people died as a result of lightning than hurricanes for each year.

35. What type of severe weather accounted for the most deaths each year?

Online Research **Data Update** What are the current weather statistics? Visit www.algebra2.com/data_update to learn more.

RECREATION For Exercises 36–39, use the following price list for one-day admissions to the community pool.

36. Write a matrix that represents the cost of admission for residents and a matrix that represents the cost of admission for nonresidents.

37. Find the matrix that represents the additional cost for nonresidents.

38. Write a matrix that represents the cost of admission before 6:00 P.M. and a matrix that represents the cost of admission after 6:00 P.M.

39. Find a matrix that represents the difference in cost if a child or adult goes to the pool after 6:00 P.M.

Daily Admission Fees

Residents

Time of day	Child	Adult
Before 6:00 P.M.	$3.00	$4.50
After 6:00 P.M.	$2.00	$3.50

Nonresidents

Time of day	Child	Adult
Before 6:00 P.M.	$4.50	$6.75
After 6:00 P.M.	$3.00	$5.25

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40. CRITICAL THINKING Determine values for each variable if $d = 1$, $e = 4d$, $z + d = e$, $f = \frac{x}{5}$, $ay = 1.5$, $x = \frac{d}{2}$, and $y = x + \frac{x}{2}$.

$$a\begin{bmatrix} x & y & z \\ d & e & f \end{bmatrix} = \begin{bmatrix} ax & ay & az \\ ad & ae & af \end{bmatrix}$$

41. WRITING IN MATH Answer the question that was posed at the beginning of the lesson.

How can matrices be used to calculate daily dietary needs?

Include the following in your answer:
- three matrices that represent breakfast, lunch, and dinner over the three-day period, and
- a matrix that represents the total Calories, protein, and fat consumed each day.

Standardized
Test Practice
Ⓐ Ⓑ Ⓒ Ⓓ

42. Which matrix equals $\begin{bmatrix} 5 & -2 \\ -3 & 7 \end{bmatrix} - \begin{bmatrix} 3 & 4 \\ -5 & 6 \end{bmatrix}$?

Ⓐ $\begin{bmatrix} 2 & 2 \\ -8 & 1 \end{bmatrix}$ Ⓑ $\begin{bmatrix} 8 & -6 \\ -8 & 1 \end{bmatrix}$ Ⓒ $\begin{bmatrix} 2 & 2 \\ 2 & 1 \end{bmatrix}$ Ⓓ $\begin{bmatrix} 2 & -6 \\ 2 & 1 \end{bmatrix}$

43. Solve for x and y in the matrix equation $\begin{bmatrix} x \\ 7 \end{bmatrix} + \begin{bmatrix} 3y \\ -x \end{bmatrix} = \begin{bmatrix} 16 \\ 12 \end{bmatrix}$.

Ⓐ $(-5, 7)$ Ⓑ $(7, 5)$ Ⓒ $(7, 3)$ Ⓓ $(5, 7)$

Maintain Your Skills

Mixed Review State the dimensions of each matrix. *(Lesson 4-1)*

44. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

45. $[2 \quad 0 \quad 3 \quad 0]$

46. $\begin{bmatrix} 5 & 1 & -6 & 2 \\ -38 & 5 & 7 & 3 \end{bmatrix}$

47. $\begin{bmatrix} 7 & -3 & 5 \\ 0 & 2 & -9 \\ 6 & 5 & 1 \end{bmatrix}$

48. $\begin{bmatrix} 8 & 6 \\ 5 & 2 \\ -4 & -1 \end{bmatrix}$

49. $\begin{bmatrix} 7 & 5 & 0 \\ -8 & 3 & 8 \\ 9 & -1 & 15 \\ 4 & 2 & 11 \end{bmatrix}$

Solve each system of equations. *(Lesson 3-5)*

50. $2a + b = 2$
$5a = 15$
$a + b + c = -1$

51. $r + s + t = 15$
$r + t = 12$
$s + t = 10$

52. $6x - 2y - 3z = -10$
$-6x + y + 9z = 3$
$8x - 3y = -16$

Solve each system by using substitution or elimination. *(Lesson 3-2)*

53. $2s + 7t = 39$
$5s - t = 5$

54. $3p + 6q = -3$
$2p - 3q = -9$

55. $a + 5b = 1$
$7a - 2b = 44$

SCRAPBOOKS For Exercises 56–58, use the following information.
Ian has $6.00, and he wants to buy paper for his scrapbook. A sheet of printed paper costs 30¢, and a sheet of solid color paper costs 15¢. *(Lesson 2-7)*

56. Write an inequality that describes this situation.

57. Graph the inequality.

58. Does Ian have enough money to buy 14 pieces of each type of paper?

Getting Ready for the Next Lesson **PREREQUISITE SKILL** Name the property illustrated by each equation.
*(To review the **properties of equality**, see Lesson 1-2.)*

59. $\frac{7}{9} \cdot \frac{9}{7} = 1$

60. $7 + (w + 5) = (7 + w) + 5$

61. $3(x + 12) = 3x + 3(12)$

62. $6(9a) = 9a(6)$

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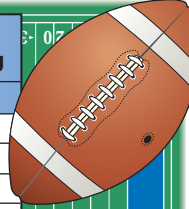
Multiplying Matrices

What You'll Learn

- Multiply matrices.
- Use the properties of matrix multiplication.

How can matrices be used in sports statistics?

Professional football teams track many statistics throughout the season to help evaluate their performance. The table shows the scoring summary of the Oakland Raiders for the 2000 season. The team's record can be summarized in the record matrix R. The values for each type of score can be organized in the point values matrix P.

Oakland Raiders Regular Season Scoring	
Type	**Number**
Touchdown	58
Extra Point	56
Field Goal	23
2-Point Conversion	1
Safety	2

Source: National Football League

Record

$$R = \begin{bmatrix} 58 \\ 56 \\ 23 \\ 1 \\ 2 \end{bmatrix} \begin{matrix} \text{touchdown} \\ \text{extra point} \\ \text{field goal} \\ \text{2-point conversion} \\ \text{safety} \end{matrix}$$

Point Values

$$P = [6 \quad 1 \quad 3 \quad 2 \quad 2]$$

touchdown, extra point, field goal, 2-point conversion, safety

You can use matrix multiplication to find the total points scored.

MULTIPLY MATRICES You can multiply two matrices if and only if the number of columns in the first matrix is equal to the number of rows in the second matrix. When you multiply two matrices $A_{m \times n}$ and $B_{n \times r}$, the resulting matrix AB is an $m \times r$ matrix.

outer dimensions = dimensions of AB

$$A \cdot B = AB$$
$$2 \times 3 \quad 3 \times 4 \qquad 2 \times 4$$

inner dimensions are equal
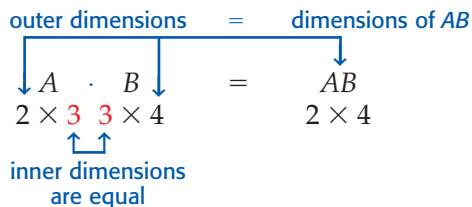
Example 1 Dimensions of Matrix Products

Determine whether each matrix product is defined. If so, state the dimensions of the product.

a. $A_{2 \times 5}$ and $B_{5 \times 4}$

$$A \cdot B = AB$$
$$2 \times 5 \quad 5 \times 4 \quad 2 \times 4$$

The inner dimensions are equal so the matrix product is defined. The dimensions of the product are 2×4.

b. $A_{1 \times 3}$ and $B_{4 \times 3}$

$$A \cdot B$$
$$1 \times 3 \quad 4 \times 3$$

The inner dimensions are not equal, so the matrix product is not defined.

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The product of two matrices is found by multiplying columns and rows. The entry in the first row and first column of AB is found by multiplying corresponding elements in the first row of A and the first column of B and then adding.

Key Concept   **Multiply Matrices**

- **Words**   The element a_{ij} of AB is the sum of the products of the corresponding elements in row i of A and column j of B.

- **Symbols** $\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \cdot \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix} = \begin{bmatrix} a_1x_1 + b_1x_2 & a_1y_1 + b_1y_2 \\ a_2x_1 + b_2x_2 & a_2y_1 + b_2y_2 \end{bmatrix}$

Example 2 *Multiply Square Matrices*

Find RS if $R = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}$ and $S = \begin{bmatrix} 3 & -9 \\ 5 & 7 \end{bmatrix}$.

$RS = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 3 & -9 \\ 5 & 7 \end{bmatrix}$

Step 1 Multiply the numbers in the first row of R by the numbers in the first column of S, add the products, and put the result in the first row, first column of RS.

$\begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 3 & -9 \\ 5 & 7 \end{bmatrix} = \begin{bmatrix} 2(3) + (-1)(5) & \\ & \end{bmatrix}$

Step 2 Multiply the numbers in the first row of R by the numbers in the second column of S, add the products, and put the result in the first row, second column of RS.

$\begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 3 & -9 \\ 5 & 7 \end{bmatrix} = \begin{bmatrix} 2(3) + (-1)(5) & 2(-9) + (-1)(7) \\ & \end{bmatrix}$

Step 3 Multiply the numbers in the second row of R by the numbers in the first column of S, add the products, and put the result in the second row, first column of RS.

$\begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 3 & -9 \\ 5 & 7 \end{bmatrix} = \begin{bmatrix} 2(3) + (-1)(5) & 2(-9) + (-1)(7) \\ 3(3) + 4(5) & \end{bmatrix}$

Step 4 Multiply the numbers in the second row of R by the numbers in the second column of S, add the products, and put the result in the second row, second column of RS.

$\begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 3 & -9 \\ 5 & 7 \end{bmatrix} = \begin{bmatrix} 2(3) + (-1)(5) & 2(-9) + (-1)(7) \\ 3(3) + 4(5) & 3(-9) + 4(7) \end{bmatrix}$

Step 5 Simplify the product matrix.

$\begin{bmatrix} 2(3) + (-1)(5) & 2(-9) + (-1)(7) \\ 3(3) + 4(5) & 3(-9) + 4(7) \end{bmatrix} = \begin{bmatrix} 1 & -25 \\ 29 & 1 \end{bmatrix}$

So, $RS = \begin{bmatrix} 1 & -25 \\ 29 & 1 \end{bmatrix}$.

Study Tip

Multiplying Matrices

To avoid any miscalculations, find the product of the matrices in order as shown in Example 2. It may also help to cover rows or columns not being multiplied as you find elements of the product matrix.

When solving real-world problems, make sure to multiply the matrices in the order for which the product is defined.

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Example **3** *Multiply Matrices with Different Dimensions*

TRACK AND FIELD In a four-team track meet, 5 points were awarded for each first-place finish, 3 points for each second, and 1 point for each third. Find the total number of points for each school. Which school won the meet?

School	First Place	Second Place	Third Place
Jefferson	8	4	5
London	6	3	7
Springfield	5	7	3
Madison	7	5	4

Explore   The final scores can be found by multiplying the track results for each school by the points awarded for each first-, second-, and third-place finish.

Plan   Write the results of the races and the points awarded in matrix form. Set up the matrices so that the number of rows in the points matrix equals the number of columns in the results matrix.

$$
\textbf{Results} \qquad\qquad \textbf{Points}
$$

$$
R = \begin{bmatrix} 8 & 4 & 5 \\ 6 & 3 & 7 \\ 5 & 7 & 3 \\ 7 & 5 & 4 \end{bmatrix} \qquad\qquad P = \begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix}
$$

Solve   Multiply the matrices.

$$
RP = \begin{bmatrix} 8 & 4 & 5 \\ 6 & 3 & 7 \\ 5 & 7 & 3 \\ 7 & 5 & 4 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix} \qquad \textcolor{blue}{\text{Write an equation.}}
$$

$$
= \begin{bmatrix} 8(5) + 4(3) + 5(1) \\ 6(5) + 3(3) + 7(1) \\ 5(5) + 7(3) + 3(1) \\ 7(5) + 5(3) + 4(1) \end{bmatrix} \qquad \textcolor{blue}{\text{Multiply columns by rows.}}
$$

$$
= \begin{bmatrix} 57 \\ 46 \\ 49 \\ 54 \end{bmatrix} \qquad \textcolor{blue}{\text{Simplify.}}
$$

The labels for the product matrix are shown below.

$$
\textbf{Total Points}
$$

$$
\begin{matrix} \text{Jefferson} \\ \text{London} \\ \text{Springfield} \\ \text{Madison} \end{matrix} \begin{bmatrix} 57 \\ 46 \\ 49 \\ 54 \end{bmatrix}
$$

Jefferson won the track meet with a total of 57 points.

Examine R is a 4×3 matrix and P is a 3×1 matrix; so their product should be a 4×1 matrix.   *Why?*

MULTIPLICATIVE PROPERTIES   Recall that the same properties for real numbers also held true for matrix addition. However, some of these properties do *not* always hold true for matrix multiplication.

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Example 4 **Commutative Property**

Find each product if $P = \begin{bmatrix} 8 & -7 \\ -2 & 4 \\ 0 & 3 \end{bmatrix}$ and $Q = \begin{bmatrix} 9 & -3 & 2 \\ 6 & -1 & -5 \end{bmatrix}$.

a. PQ

$PQ = \begin{bmatrix} 8 & -7 \\ -2 & 4 \\ 0 & 3 \end{bmatrix} \cdot \begin{bmatrix} 9 & -3 & 2 \\ 6 & -1 & -5 \end{bmatrix}$ Substitution

$= \begin{bmatrix} 72 - 42 & -24 + 7 & 16 + 35 \\ -18 + 24 & 6 - 4 & -4 - 20 \\ 0 + 18 & 0 - 3 & 0 - 15 \end{bmatrix}$ Multiply columns by rows.

$= \begin{bmatrix} 30 & -17 & 51 \\ 6 & 2 & -24 \\ 18 & -3 & -15 \end{bmatrix}$ Simplify.

b. QP

$QP = \begin{bmatrix} 9 & -3 & 2 \\ 6 & -1 & -5 \end{bmatrix} \cdot \begin{bmatrix} 8 & -7 \\ -2 & 4 \\ 0 & 3 \end{bmatrix}$ Substitution

$= \begin{bmatrix} 72 + 6 + 0 & -63 - 12 + 6 \\ 48 + 2 + 0 & -42 - 4 - 15 \end{bmatrix}$ Multiply columns by rows.

$= \begin{bmatrix} 78 & -69 \\ 50 & -61 \end{bmatrix}$ Simplify.

In Example 4, notice that $PQ \neq QP$ because $\begin{bmatrix} 30 & -17 & 51 \\ 6 & 2 & -24 \\ 18 & -3 & -15 \end{bmatrix} \neq \begin{bmatrix} 78 & -69 \\ 50 & -61 \end{bmatrix}$.

This demonstrates that the Commutative Property of Multiplication does not hold for matrix multiplication. The order in which you multiply matrices is very important.

Example 5 **Distributive Property**

Find each product if $A = \begin{bmatrix} 3 & 2 \\ -1 & 4 \end{bmatrix}$, $B = \begin{bmatrix} -2 & 5 \\ 6 & 7 \end{bmatrix}$, and $C = \begin{bmatrix} 1 & 1 \\ -5 & 3 \end{bmatrix}$.

a. $A(B + C)$

$A(B + C) = \begin{bmatrix} 3 & 2 \\ -1 & 4 \end{bmatrix} \cdot \left(\begin{bmatrix} -2 & 5 \\ 6 & 7 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ -5 & 3 \end{bmatrix} \right)$ Substitution

$= \begin{bmatrix} 3 & 2 \\ -1 & 4 \end{bmatrix} \cdot \begin{bmatrix} -1 & 6 \\ 1 & 10 \end{bmatrix}$ Add corresponding elements.

$= \begin{bmatrix} 3(-1) + 2(1) & 3(6) + 2(10) \\ -1(-1) + 4(1) & -1(6) + 4(10) \end{bmatrix}$ or $\begin{bmatrix} -1 & 38 \\ 5 & 34 \end{bmatrix}$ Multiply columns by rows.

b. $AB + AC$

$AB + AC = \begin{bmatrix} 3 & 2 \\ -1 & 4 \end{bmatrix} \cdot \begin{bmatrix} -2 & 5 \\ 6 & 7 \end{bmatrix} + \begin{bmatrix} 3 & 2 \\ -1 & 4 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ -5 & 3 \end{bmatrix}$ Substitution

$= \begin{bmatrix} 3(-2) + 2(6) & 3(5) + 2(7) \\ -1(-2) + 4(6) & -1(5) + 4(7) \end{bmatrix} + \begin{bmatrix} 3(1) + 2(-5) & 3(1) + 2(3) \\ -1(1) + 4(-5) & -1(1) + 4(3) \end{bmatrix}$

$= \begin{bmatrix} 6 & 29 \\ 26 & 23 \end{bmatrix} + \begin{bmatrix} -7 & 9 \\ -21 & 11 \end{bmatrix}$ Simplify.

$= \begin{bmatrix} -1 & 38 \\ 5 & 34 \end{bmatrix}$ Add corresponding elements.

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Notice that in Example 5, $A(B + C) = AB + AC$. This and other examples suggest that the Distributive Property is true for matrix multiplication. Some properties of matrix multiplication are shown below.

Concept Summary **Properties of Matrix Multiplication**

For any matrices A, B, and C for which the matrix product is defined, and any scalar c, the following properties are true.

Associative Property of Matrix Multiplication $(AB)C = A(BC)$

Associative Property of Scalar Multiplication $c(AB) = (cA)B = A(cB)$

Left Distributive Property $C(A + B) = CA + CB$

Right Distributive Property $(A + B)C = AC + BC$

To show that a property is true for all cases, you must show it is true for the general case. To show that a property is *not* true for all cases, you only need to find a counterexample.

Check for Understanding

Concept Check **1. OPEN ENDED** Give an example of two matrices whose product is a 3×2 matrix.

2. **Determine** whether the following statement is *always*, *sometimes*, or *never* true. Explain your reasoning.

   For any matrix $A_{m \times n}$ for $m \neq n$, A^2 is defined.

3. **Explain** why, in most cases, $(A + B)C \neq CA + CB$.

Guided Practice **Determine whether each matrix product is defined. If so, state the dimensions of the product.**

4. $A_{3 \times 5} \cdot B_{5 \times 2}$

5. $X_{2 \times 3} \cdot Y_{2 \times 3}$

Find each product, if possible.

6. $[3 \ -5] \cdot \begin{bmatrix} 3 & 5 \\ -2 & 0 \end{bmatrix}$

7. $\begin{bmatrix} 5 \\ 8 \end{bmatrix} \cdot [3 \ -1 \ 4]$

8. $\begin{bmatrix} 5 & -2 & -1 \\ 8 & 0 & 3 \end{bmatrix} \cdot \begin{bmatrix} -4 & 2 \\ 1 & 0 \end{bmatrix}$

9. $\begin{bmatrix} 4 & -1 \\ 3 & 5 \end{bmatrix} \cdot \begin{bmatrix} 7 \\ 4 \end{bmatrix}$

10. Use $A = \begin{bmatrix} 2 & -1 \\ 3 & 5 \end{bmatrix}$, $B = \begin{bmatrix} -4 & 1 \\ 8 & 0 \end{bmatrix}$, and $C = \begin{bmatrix} 3 & 2 \\ -1 & 2 \end{bmatrix}$ to determine whether $A(BC) = (AB)C$ is true for the given matrices.

Application **SPORTS** For Exercises 11 and 12, use the table below that shows the number of kids registered for baseball and softball.

The Westfall Youth Baseball and Softball League charges the following registration fees: ages 7–8, $45; ages 9–10, $55; and ages 11–14, $65.

11. Write a matrix for the registration fees and a matrix for the number of players.

12. Find the total amount of money the League received from baseball and softball registrations.

Team Members		
Age	Baseball	Softball
7–8	350	280
9–10	320	165
11–14	180	120

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Practice and Apply

Extra Practice
See page 834.

Determine whether each matrix product is defined. If so, state the dimensions of the product.

13. $A_{4 \times 3} \cdot B_{3 \times 2}$ **14.** $X_{2 \times 2} \cdot Y_{2 \times 2}$ **15.** $P_{1 \times 3} \cdot Q_{4 \times 1}$

16. $R_{1 \times 4} \cdot S_{4 \times 5}$ **17.** $M_{4 \times 3} \cdot N_{4 \times 3}$ **18.** $A_{3 \times 1} \cdot B_{1 \times 5}$

Find each product, if possible.

19. $[2 \quad -1] \cdot \begin{bmatrix} 5 \\ 4 \end{bmatrix}$

20. $\begin{bmatrix} 3 & -2 \\ 5 & 1 \end{bmatrix} \cdot \begin{bmatrix} 4 & 1 \\ 2 & 7 \end{bmatrix}$

21. $\begin{bmatrix} 4 & -1 & 6 \\ 1 & 5 & -8 \end{bmatrix} \cdot \begin{bmatrix} 1 & 3 \\ 9 & -6 \end{bmatrix}$

22. $\begin{bmatrix} 4 & -2 & -7 \\ 6 & 3 & 5 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 5 \\ 3 \end{bmatrix}$

23. $\begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 3 & -9 & -2 \\ 5 & 7 & -6 \end{bmatrix}$

24. $\begin{bmatrix} 7 & 3 \\ 0 & 2 \\ 5 & 5 \end{bmatrix} \cdot \begin{bmatrix} -2 & 1 & 4 \\ 3 & -5 & 2 \\ 4 & 3 & 1 \end{bmatrix}$

25. $\begin{bmatrix} 4 & 0 \\ -3 & 7 \\ -5 & 9 \end{bmatrix} \cdot \begin{bmatrix} 6 & 4 \\ -2 & 1 \end{bmatrix}$

26. $\begin{bmatrix} 0 & 8 \\ 3 & 1 \\ -1 & 5 \end{bmatrix} \cdot \begin{bmatrix} 3 & 1 & -2 \\ 0 & 8 & -5 \end{bmatrix}$

Use $A = \begin{bmatrix} 1 & -2 \\ 4 & 3 \end{bmatrix}$, $B = \begin{bmatrix} -5 & 2 \\ 4 & 3 \end{bmatrix}$, $C = \begin{bmatrix} 5 & 1 \\ 2 & -4 \end{bmatrix}$, and scalar $c = 3$ to determine whether the following equations are true for the given matrices.

27. $AC + BC = (A + B)C$ **28.** $c(AB) = A(cB)$

29. $C(A + B) = AC + BC$ **30.** $ABC = CBA$

PRODUCE For Exercises 31–34, use the table and the following information.
Carmen Fox owns three fruit farms on which he grows apples, peaches, and apricots. He sells apples for $22 a case, peaches for $25 a case, and apricots for $18 a case.

Number of Cases in Stock of Each Type of Fruit			
Farm	Apples	Peaches	Apricots
1	290	165	210
2	175	240	190
3	110	75	0

31. Write an inventory matrix for the number of cases for each type of fruit for each farm.

32. Write a cost matrix for the price per case for each type of fruit.

33. Find the total income of the three fruit farms expressed as a matrix.

34. What is the total income from all three fruit farms combined?

35. CRITICAL THINKING Give an example of two matrices A and B whose product is commutative so that $AB = BA$.

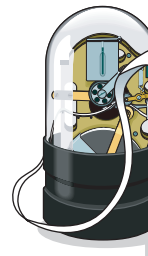**FUND-RAISING** For Exercises 36–39, use the table and the information below.
Lawrence High School sold wrapping paper and boxed cards for their fund-raising event. The school receives $1.00 for each roll of wrapping paper sold and $0.50 for each box of cards sold.

Total Amounts for Each Class		
Class	Wrapping Paper	Cards
Freshmen	72	49
Sophomores	68	63
Juniors	90	56
Seniors	86	62

36. Write a matrix that represents the amounts sold for each class and a matrix that represents the amount of money the school earns for each item sold.

37. Write a matrix that shows how much each class earned.

38. Which class earned the most money?

39. What is the total amount of money the school made from the fund-raiser?

FINANCE For Exercises 40–42, use the table below that shows the purchase price and selling price of stock for three companies.

For a class project, Taini "bought" shares of stock in three companies. She bought 150 shares of a utility company, 100 shares of a computer company, and 200 shares of a food company. At the end of the project she "sold" all of her stock.

Company	Purchase Price (per share)	Selling Price (per share)
Utility	$54.00	$55.20
Computer	$48.00	$58.60
Food	$60.00	$61.10

40. Organize the data in two matrices and use matrix multiplication to find the total amount she spent for the stock.

41. Write two matrices and use matrix multiplication to find the total amount she received for selling the stock.

42. Use matrix operations to find how much money Taini "made" or "lost."

43. CRITICAL THINKING Find the values of a, b, c, and d to make the statement $\begin{bmatrix} 3 & 5 \\ -1 & 7 \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ -1 & 7 \end{bmatrix}$ true. If the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is multiplied by any other matrix containing two columns, what do you think the result would be?

44. WRITING IN MATH Answer the question that was posed at the beginning of the lesson.

How can matrices be used in sports statistics?

Include the following in your answer:
• a matrix that represents the total points scored in the 2000 season, and
• an example of another sport where different point values are used in scoring.

45. If C is a 5×1 matrix and D is a 3×5 matrix, what are the dimensions of DC?
   Ⓐ 5×5 Ⓑ 3×1 Ⓒ 1×3 Ⓓ DC is not defined.

46. What is the product of $\begin{bmatrix} 5 & -2 & 3 \end{bmatrix}$ and $\begin{bmatrix} 1 & -2 \\ 0 & 3 \\ 2 & 5 \end{bmatrix}$?

   Ⓐ $\begin{bmatrix} 11 & -1 \end{bmatrix}$ Ⓑ $\begin{bmatrix} 11 \\ -1 \end{bmatrix}$ Ⓒ $\begin{bmatrix} 5 & -10 \\ 0 & -6 \\ 6 & -15 \end{bmatrix}$ Ⓓ undefined

www.algebra2.com/self_check_quiz

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Mixed Review Perform the indicated matrix operations. If the matrix does not exist, write impossible. *(Lesson 4-2)*

47. $3\begin{bmatrix} 4 & -2 \\ -1 & 7 \end{bmatrix}$

48. $[3 \quad 5 \quad 9] + \begin{bmatrix} 5 \\ 2 \\ 6 \end{bmatrix}$

49. $2\begin{bmatrix} 6 & 3 \\ -8 & -2 \end{bmatrix} - 4\begin{bmatrix} 8 & 1 \\ 3 & -4 \end{bmatrix}$

Solve each equation. *(Lesson 4-1)*

50. $\begin{bmatrix} 3x + 2 \\ 15 \end{bmatrix} = \begin{bmatrix} 23 \\ -4y - 1 \end{bmatrix}$

51. $\begin{bmatrix} x + 3y \\ 2x - y \end{bmatrix} = \begin{bmatrix} -22 \\ 19 \end{bmatrix}$

52. $\begin{bmatrix} x + 3z \\ -2x + y - z \\ 5y - 7z \end{bmatrix} = \begin{bmatrix} -19 \\ -2 \\ 24 \end{bmatrix}$

53. CAMERA SUPPLIES Mrs. Franklin is planning a family vacation. She bought 8 rolls of film and 2 camera batteries for $23. The next day, her daughter went back and bought 6 more rolls of film and 2 batteries for her camera. This bill was $18. What is the price of a roll of film and a camera battery? *(Lesson 3-2)*

Find the *x*-intercept and the *y*-intercept of the graph of each equation. Then graph the equation. *(Lesson 2-2)*

54. $y = 3 - 2x$

55. $x - \frac{1}{2}y = 8$

56. $5x - 2y = 10$

Getting Ready for the Next Lesson **PREREQUISITE SKILL** Graph each set of ordered pairs on a coordinate plane.
*(To review **graphing ordered pairs**, see Lesson 2-1.)*

57. $\{(2, 4), (-1, 3), (0, -2)\}$

58. $\{(-3, 5), (-2, -4), (3, -2)\}$

59. $\{(-1, 2), (2, 4), (3, -3), (4, -1)\}$

60. $\{(-3, 3), (1, 3), (4, 2), (-1, -5)\}$

Practice Quiz 1
Lessons 4-1 through 4-3

Solve each equation. *(Lesson 4-1)*

1. $\begin{bmatrix} 3x + 1 \\ 7y \end{bmatrix} = \begin{bmatrix} 19 \\ 21 \end{bmatrix}$

2. $\begin{bmatrix} 2x + y \\ 4x - 3y \end{bmatrix} = \begin{bmatrix} 9 \\ 23 \end{bmatrix}$

3. $\begin{bmatrix} 2 & x \\ y & 5 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 3 & z \end{bmatrix}$

BUSINESS For Exercises 4 and 5, use the table and the following information.
The manager of The Best Bagel Shop keeps records of each type of bagel sold each day at their two stores. Two days of sales are shown below.

Day	Store	Type of Bagel			
		Sesame	**Poppy**	**Blueberry**	**Plain**
Monday	East	120	80	64	75
	West	65	105	77	53
Tuesday	East	112	79	56	74
	West	69	95	82	50

4. Write a matrix for each day's sales. *(Lesson 4-1)*
5. Find the sum of the two days' sales using matrix addition. *(Lesson 4-2)*

Perform the indicated matrix operations. *(Lesson 4-2)*

6. $\begin{bmatrix} 3 & 0 \\ 7 & 12 \end{bmatrix} - \begin{bmatrix} 6 & -5 \\ 4 & -1 \end{bmatrix}$

7. $\frac{2}{3}\begin{bmatrix} 9 & 0 \\ 12 & 15 \end{bmatrix} + \begin{bmatrix} -2 & 3 \\ -7 & -7 \end{bmatrix}$

8. $5\begin{bmatrix} -2 & 4 & 5 \\ 0 & -4 & 7 \end{bmatrix}$

Find each product, if possible. *(Lesson 4-3)*

9. $\begin{bmatrix} 4 & 0 & -8 \\ 7 & -2 & 10 \end{bmatrix} \cdot \begin{bmatrix} -1 & 3 \\ 6 & 0 \end{bmatrix}$

10. $\begin{bmatrix} 3 & -1 \\ 2 & 5 \end{bmatrix} \cdot \begin{bmatrix} 4 & -1 & -2 \\ -3 & 5 & 4 \end{bmatrix}$

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Transformations with Matrices

What You'll Learn

- Use matrices to determine the coordinates of a translated or dilated figure.
- Use matrix multiplication to find the coordinates of a reflected or rotated figure.

Vocabulary

- vertex matrix
- transformation
- preimage
- image
- isometry
- translation
- dilation
- reflection
- rotation

How are transformations used in computer animation?

Computer animation creates the illusion of motion by using a succession of computer-generated still images. Computer animation is used to create movie special effects and to simulate images that would be impossible to show otherwise. An object's size and orientation are stored in a computer program. Complex geometric figures can be broken into simple triangles and then moved to other parts of the screen.

TRANSLATIONS AND DILATIONS Points on a coordinate plane can be represented by matrices. The ordered pair (x, y) can be represented by the column matrix $\begin{bmatrix} x \\ y \end{bmatrix}$. Likewise, polygons can be represented by placing all of the column matrices of the coordinates of the vertices into one matrix, called a **vertex matrix**.

Triangle ABC with vertices $A(3, 2)$, $B(4, -2)$, and $C(2, -1)$ can be represented by the following vertex matrix.

$$\triangle ABC = \begin{matrix} A & B & C \\ \begin{bmatrix} 3 & 4 & 2 \\ 2 & -2 & -1 \end{bmatrix} & & \end{matrix} \begin{matrix} \leftarrow x\text{-coordinates} \\ \leftarrow y\text{-coordinates} \end{matrix}$$
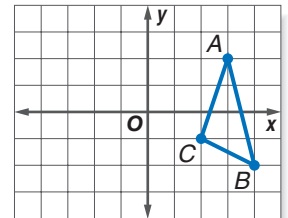
One of the ways that matrices are used is to perform transformations. **Transformations** are functions that map points of a **preimage** onto its **image**. If the image and preimage are congruent figures, the transformation is an **isometry**.

One type of isometry is a translation. A **translation** occurs when a figure is moved from one location to another without changing its size, shape, or orientation. You can use matrix addition and a *translation matrix* to find the coordinates of a translated figure.

Example 1 Translate a Figure

Find the coordinates of the vertices of the image of quadrilateral *QUAD* with $Q(2, 3)$, $U(5, 2)$, $A(4, -2)$, and $D(1, -1)$, if it is moved 4 units to the left and 2 units up. Then graph *QUAD* and its image $Q'U'A'D'$.

Write the vertex matrix for quadrilateral *QUAD*. $\begin{bmatrix} 2 & 5 & 4 & 1 \\ 3 & 2 & -2 & -1 \end{bmatrix}$

To translate the quadrilateral 4 units to the left, add -4 to each x-coordinate. To translate the figure 2 units up, add 2 to each y-coordinate. This can be done by adding the translation matrix $\begin{bmatrix} -4 & -4 & -4 & -4 \\ 2 & 2 & 2 & 2 \end{bmatrix}$ to the vertex matrix of *QUAD*.

Study Tip

Reading Math
A matrix containing coordinates of a geometric figure is also called a *coordinate matrix*.

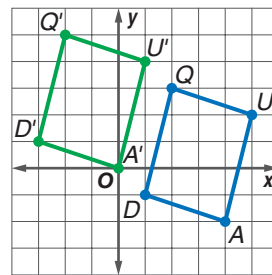| Vertex Matrix of *QUAD*				Translation Matrix				Vertex Matrix of *Q′U′A′D′*			

$$\begin{bmatrix} 2 & 5 & 4 & 1 \\ 3 & 2 & -2 & -1 \end{bmatrix} + \begin{bmatrix} -4 & -4 & -4 & -4 \\ 2 & 2 & 2 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 & -3 \\ 5 & 4 & 0 & 1 \end{bmatrix}$$

The coordinates of $Q′U′A′D′$ are $Q′(-2, 5)$, $U′(1, 4)$, $A′(0, 0)$, and $D′(-3, 1)$. Graph the preimage and the image. The two quadrilaterals have the same size and shape.

Standardized Test Practice

Ⓐ Ⓑ Ⓒ Ⓓ

Example 2 Find a Translation Matrix

Short-Response Test Item

Rectangle $A′B′C′D′$ is the result of a translation of rectangle $ABCD$. A table of the vertices of each rectangle is shown. Find the coordinates of A and $D′$.

Rectangle ABCD	Rectangle A′B′C′D′
A	A′(−1, 1)
B(1, 5)	B′(4, 1)
C(1, −2)	C′(4, −6)
D(−4, −2)	D′

Read the Test Item

- You are given the coordinates of the preimage and image of points B and C. Use this information to find the translation matrix. Then you can use the translation matrix to find the coordinates of A and $D′$.

The Princeton Review

Test-Taking Tip

Sometimes you need to solve for unknown value(s) before you can solve for the value(s) requested in the question.

Solve the Test Item

- Write a matrix equation. Let (a, b) represent the coordinates of A and let (c, d) represent the coordinates of $D′$.

$$\begin{bmatrix} a & 1 & 1 & -4 \\ b & 5 & -2 & -2 \end{bmatrix} + \begin{bmatrix} x & x & x & x \\ y & y & y & y \end{bmatrix} = \begin{bmatrix} -1 & 4 & 4 & c \\ 1 & 1 & -6 & d \end{bmatrix}$$

$$\begin{bmatrix} a+x & 1+x & 1+x & -4+x \\ b+y & 5+y & -2+y & -2+y \end{bmatrix} = \begin{bmatrix} -1 & 4 & 4 & c \\ 1 & 1 & -6 & d \end{bmatrix}$$

- Since these two matrices are equal, corresponding elements are equal.

  Solve an equation for x.

  $1 + x = 4$
  $\quad x = 3$

  Solve an equation for y.

  $5 + y = 1$
  $\quad y = -4$

- Use the values for x and y to find the values for $A(a, b)$ and $D′(c, d)$.

  $a + x = -1 \qquad\qquad b + y = 1 \qquad\qquad -4 + x = c \qquad\qquad -2 + y = d$
  $a + 3 = -1 \qquad\qquad b + (-4) = 1 \qquad\quad -4 + 3 = c \qquad\qquad -2 + (-4) = d$
  $\quad a = -4 \qquad\qquad\qquad b = 5 \qquad\qquad\qquad -1 = c \qquad\qquad\qquad -6 = d$

So the coordinates of A are $(-4, 5)$, and the coordinates for $D′$ are $(-1, -6)$.

When a geometric figure is enlarged or reduced, the transformation is called a **dilation**. In a dilation, all linear measures of the image change in the same ratio. For example, if the length of each side of a figure doubles, then the perimeter doubles, and vice versa. You can use scalar multiplication to perform dilations.

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Example **3** *Dilation*

△*JKL* has vertices *J*(−2, −3), *K*(−5, 4), and *L*(3, 2). Dilate △*JKL* so that its perimeter is one-half the original perimeter. What are the coordinates of the vertices of △*J′K′L′*?

If the perimeter of a figure is one-half the original perimeter, then the lengths of the sides of the figure will be one-half the measure of the original lengths. Multiply the vertex matrix by the scale factor of $\frac{1}{2}$.

$$\frac{1}{2}\begin{bmatrix} -2 & -5 & 3 \\ -3 & 4 & 2 \end{bmatrix} = \begin{bmatrix} -1 & -\frac{5}{2} & \frac{3}{2} \\ -\frac{3}{2} & 2 & 1 \end{bmatrix}$$

The coordinates of the vertices of △*J′K′L′* are $J'\left(-1, -\frac{3}{2}\right)$, $K'\left(-\frac{5}{2}, 2\right)$, and $L'\left(\frac{3}{2}, 1\right)$.

Graph △*JKL* and △*J′K′L′*. The triangles are not congruent. The image has sides that are half the length of those of the original figure.
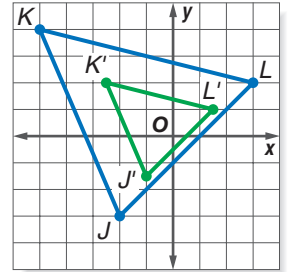
REFLECTIONS AND ROTATIONS   In addition to translations, reflections and rotations are also isometries. A **reflection** occurs when every point of a figure is mapped to a corresponding image across a line of symmetry using a *reflection matrix*. The matrices used for three common reflections are shown below.

| Concept Summary		Reflection Matrices	
For a reflection over the:	*x*-axis	*y*-axis	line *y* = *x*
Multiply the vertex matrix on the left by:	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$	$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Example **4** *Reflection*

Find the coordinates of the vertices of the image of pentagon *QRSTU* with *Q*(1, 3), *R*(3, 2), *S*(3, −1), *T*(1, −2), and *U*(−1, 1) after a reflection across the *y*-axis.

Write the ordered pairs as a vertex matrix. Then multiply the vertex matrix by the reflection matrix for the *y*-axis.

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 3 & 3 & 1 & -1 \\ 3 & 2 & -1 & -2 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -3 & -3 & -1 & 1 \\ 3 & 2 & -1 & -2 & 1 \end{bmatrix}$$

The coordinates of the vertices of *Q′R′S′T′U′* are *Q′*(−1, 3), *R′*(−3, 2), *S′*(−3, −1), *T′*(−1, −2), and *U′*(1, 1). Notice that the preimage and image are congruent. Both figures have the same size and shape.

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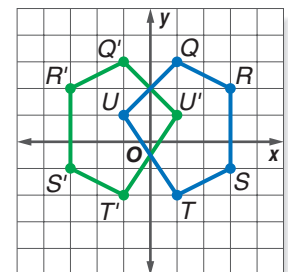
A **rotation** occurs when a figure is moved around a center point, usually the origin. To determine the vertices of a figure's image by rotation, multiply its vertex matrix by a *rotation matrix*. Commonly used rotation matrices are summarized below.

Concept Summary			Rotation Matrices
For a counterclockwise rotation about the origin of:	90°	180°	270°
Multiply the vertex matrix on the left by:	$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

Example 5 Rotation

Find the coordinates of the vertices of the image of △ABC with A(4, 3), B(2, 1), and C(1, 5) after it is rotated 90° counterclockwise about the origin.

Write the ordered pairs in a vertex matrix. Then multiply the vertex matrix by the rotation matrix.

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 4 & 2 & 1 \\ 3 & 1 & 5 \end{bmatrix} = \begin{bmatrix} -3 & -1 & -5 \\ 4 & 2 & 1 \end{bmatrix}$$

The coordinates of the vertices of △A'B'C' are A'(−3, 4), B'(−1, 2), and C'(−5, 1). The image is congruent to the preimage.
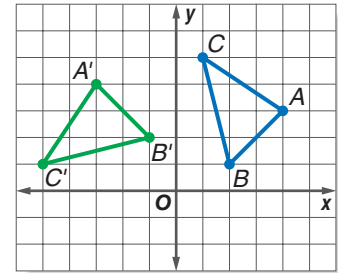
Check for Understanding

Concept Check

1. **Compare and contrast** the size and shape of the preimage and image for each type of transformation. Tell which transformations are isometries.

2. **Write** the translation matrix for △ABC and its image △A'B'C' shown at the right.

3. **OPEN ENDED** Write a translation matrix that moves △DEF up and left on the coordinate plane.
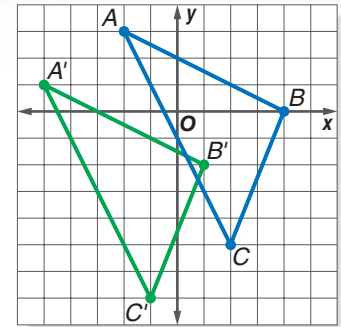
Guided Practice

Triangle ABC with vertices A(1, 4), B(2, −5), and C(−6, −6) is translated 3 units right and 1 unit down.

4. Write the translation matrix.

5. Find the coordinates of △A'B'C'.

6. Graph the preimage and the image.

For Exercises 7–10, use the rectangle at the right.

7. Write the coordinates in a vertex matrix.

8. Find the coordinates of the image after a dilation by a scale factor of 3.

9. Find the coordinates of the image after a reflection over the x-axis.

10. Find the coordinates of the image after a rotation of 180°.

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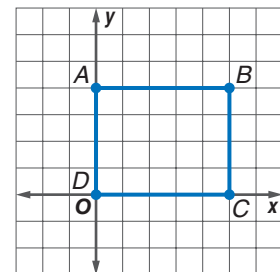
11. A point is translated from B to C as shown at the right. If a point at $(-4, 3)$ is translated in the same way, what will be its new coordinates?
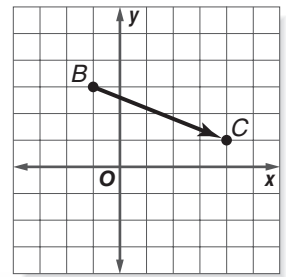
    Ⓐ $(3, 4)$ Ⓑ $(1, 1)$

    Ⓒ $(-7, 8)$ Ⓓ $(1, 6)$

Practice and Apply

For Exercises 12–14, use the following information.
Triangle DEF with vertices $D(1, 4)$, $E(2, -5)$, and $F(-6, -6)$ is translated 4 units left and 2 units up.

12. Write the translation matrix.

13. Find the coordinates of $\triangle D'E'F'$.

14. Graph the preimage and the image.

For Exercises 15–17, use the following information.
The vertices of $\triangle ABC$ are $A(0, 2)$, $B(1.5, -1.5)$, and $C(-2.5, 0)$. The triangle is dilated so that its perimeter is three times the original perimeter.

15. Write the coordinates for $\triangle ABC$ in a vertex matrix.

16. Find the coordinates of the image $\triangle A'B'C'$.

17. Graph $\triangle ABC$ and $\triangle A'B'C'$.

For Exercises 18–20, use the following information.
The vertices of $\triangle XYZ$ are $X(1, -1)$, $Y(2, -4)$, and $Z(7, -1)$. The triangle is reflected over the line $y = x$.

18. Write the coordinates of $\triangle XYZ$ in a vertex matrix.

19. Find the coordinates of $\triangle X'Y'Z'$.

20. Graph $\triangle XYZ$ and $\triangle X'Y'Z'$.

For Exercises 21–23, use the following information.
Parallelogram $DEFG$ with $D(2, 4)$, $E(5, 4)$, $F(4, 1)$, and $G(1, 1)$ is rotated 270° counterclockwise about the origin.

21. Write the coordinates of the parallelogram in a vertex matrix.

22. Find the coordinates of parallelogram $D'E'F'G'$.

23. Graph the preimage and the image.

24. Triangle DEF with vertices $D(-2, 2)$, $E(3, 5)$, and $F(5, -2)$ is translated so that D' is at $(1, -5)$. Find the coordinates of E' and F'.

25. A triangle is rotated 90° counterclockwise about the origin. The coordinates of the vertices are $J'(-3, -5)$, $K'(-2, 7)$, and $L'(1, 4)$. What were the coordinates of the triangle in its original position?

For Exercises 26–28, use quadrilateral $QRST$ shown at the right.

26. Write the vertex matrix. Multiply the vertex matrix by -1.

27. Graph the preimage and image.

28. What type of transformation does the graph represent?

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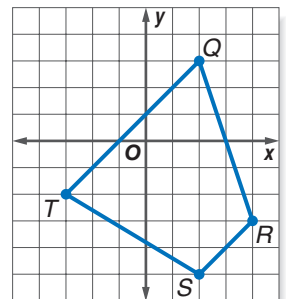
For Exercises 29–32, use rectangle *ABCD* with vertices *A*(−4, 4), *B*(4, 4), *C*(4, −4), and *D*(−4, −4).

29. Find the coordinates of the image in matrix form after a reflection over the *x*-axis followed by a reflection over the *y*-axis.

30. Find the coordinates of the image after a 180° rotation about the origin.

31. Find the coordinates of the image after a reflection over the line $y = x$.

32. What do you observe about these three matrices? Explain.

LANDSCAPING **For Exercises 33 and 34, use the following information.**
A garden design is plotted on a coordinate grid. The original plan shows a fountain with vertices at (−2, −2), (−6, −2), (−8, −5), and (−4, −5). Changes to the plan now require that the fountain's perimeter be three-fourths that of the original.

33. Determine the new coordinates for the fountain.

34. The center of the fountain was at (−5, −3.5). What will be the coordinate of the center after the changes in the plan have been made?

TECHNOLOGY **For Exercises 35 and 36, use the following information.**
As you move the mouse for your computer, a corresponding arrow is translated on the screen. Suppose the position of the cursor on the screen is given in inches with the origin at the bottom left-hand corner of the screen.

35. You want to move your cursor 3 inches to the right and 4 inches up. Write a translation matrix that can be used to move the cursor to the new position.

36. If the cursor is currently at (3.5, 2.25), what are the coordinates of the position after the translation?

37. **GYMNASTICS** The drawing at the right shows four positions of a man performing the giant swing in the high bar event. Suppose this drawing is placed on a coordinate grid with the hand grips at *H*(0, 0) and the toe of the figure in the upper right corner at *T*(7, 8). Find the coordinates of the toes of the other three figures, if each successive figure has been rotated 90° counterclockwise about the origin.

High Bar

A routine with continuous flow to quick changes in body position.

Key move:
Giant swing. As the body swings around the bar the body should be straight with a slight hollow to the chest.

Height: $8\frac{1}{2}$ feet

Length: 8 feet
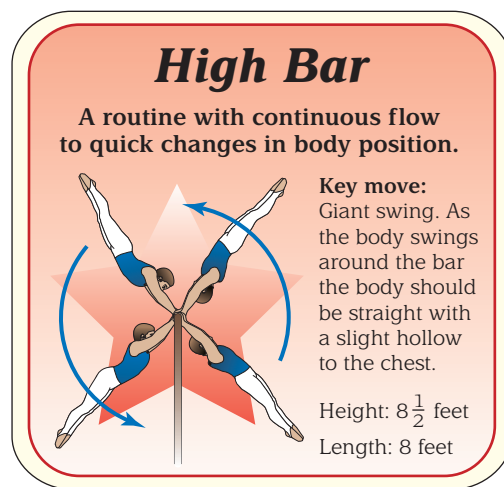
FOOTPRINTS **For Exercises 38–41, use the following information.**
The combination of a reflection and a translation is called a *glide reflection*. An example is a set of footprints.

38. Describe the reflection and transformation combination shown at the right.

39. Write two matrix operations that can be used to find the coordinates of point *C*.

40. Does it matter which operation you do first? Explain.

41. What are the coordinates of the next two footprints?
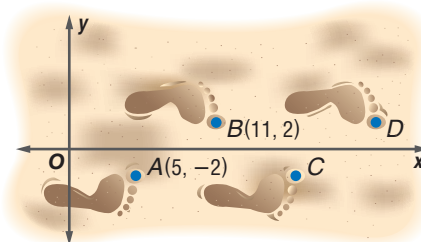
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42. **CRITICAL THINKING** Do you think a matrix exists that would represent a reflection over the line $x = 3$? If so, make a conjecture and verify it.

43. WRITING IN MATH Answer the question that was posed at the beginning of the lesson.

How are transformations used in computer animation?

Include the following in your answer:
- an example of how a figure with 5 points (coordinates) could be written in a matrix and multiplied by a rotation matrix, and
- a description of the motion that is a result of repeated dilations with a scale factor of one-fourth.

44. Which matrix represents a reflection over the y-axis followed by a reflection over the x-axis?

Ⓐ $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ Ⓑ $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ Ⓒ $\begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}$ Ⓓ none of these

45. Triangle ABC has vertices with coordinates $A(-4, 2)$, $B(-4, -3)$, and $C(3, -2)$. After a dilation, triangle $A'B'C'$ has coordinates $A'(-12, 6)$, $B'(-12, -9)$, and $C'(9, -6)$. How many times as great is the perimeter of $A'B'C'$ as ABC?

Ⓐ 3   Ⓑ 6   Ⓒ 12   Ⓓ $\frac{1}{3}$

Maintain Your Skills

Mixed Review

Determine whether each matrix product is defined. If so, state the dimensions of the product. *(Lesson 4-3)*

46. $A_{2 \times 3} \cdot B_{3 \times 2}$ 47. $A_{4 \times 1} \cdot B_{2 \times 1}$ 48. $A_{2 \times 5} \cdot B_{5 \times 5}$

Perform the indicated matrix operations. If the matrix does not exist, write **impossible**. *(Lesson 4-2)*

49. $2\begin{bmatrix} 4 & 9 & -8 \\ 6 & -11 & -2 \\ 12 & -10 & 3 \end{bmatrix} + 3\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$ 50. $4\begin{bmatrix} 3 & 4 & -7 \\ 6 & -9 & -2 \\ -3 & 1 & 3 \end{bmatrix} - \begin{bmatrix} -8 & 6 & -4 \\ -7 & 10 & 1 \\ -2 & 1 & 5 \end{bmatrix}$
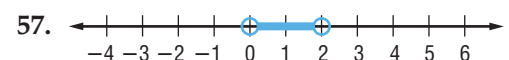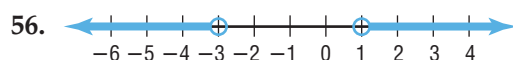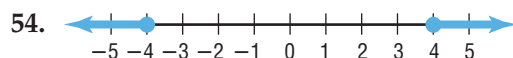
Graph each relation or equation and find the domain and range. Then determine whether the relation or equation is a function. *(Lesson 2-1)*

51. (3, 5), (4, 6), (5, −4)   52. $x = -5y + 2$ 53. $x = y^2$

Write an absolute value inequality for each graph. *(Lesson 1-6)*

54.

55.

56.

57.

58. **BUSINESS** Reliable Rentals rents cars for $12.95 per day plus 15¢ per mile. Luis Romero works for a company that limits expenses for car rentals to $90 per day. What is the maximum number of miles that Mr. Romero can drive each day? *(Lesson 1-5)*

Getting Ready for the Next Lesson

BASIC SKILL Use cross products to solve each proportion.

59. $\frac{x}{8} = \frac{3}{4}$ 60. $\frac{4}{20} = \frac{1}{m}$ 61. $\frac{2}{3} = \frac{a}{42}$

62. $\frac{5}{6} = \frac{k}{4}$ 63. $\frac{2}{y} = \frac{8}{9}$ 64. $\frac{x}{5} = \frac{x+1}{8}$

4-5 Determinants

- Evaluate the determinant of a 2 × 2 matrix.
- Evaluate the determinant of a 3 × 3 matrix.

Vocabulary

- determinant
- second-order determinant
- third-order determinant
- expansion by minors
- minor

How are determinants used to find areas of polygons?

The "Bermuda Triangle" is an area located off the southeastern Atlantic coast of the United States that is noted for a high incidence of unexplained losses of ships, small boats, and aircraft. You can estimate the area of this triangular region by finding the determinant of the matrix that contains the coordinates of the vertices of the triangle.

DETERMINANTS OF 2 × 2 MATRICES   Every square matrix has a number associated with it called its determinant. A **determinant** is a square array of numbers or variables enclosed between two parallel lines. For example, the determinant of $\begin{bmatrix} 3 & -1 \\ 2 & 5 \end{bmatrix}$ can be represented by $\begin{vmatrix} 3 & -1 \\ 2 & 5 \end{vmatrix}$ or $\det \begin{bmatrix} 3 & -1 \\ 2 & 5 \end{bmatrix}$.

The determinant of a 2 × 2 matrix is called a **second-order determinant**.

Key Concept   Second-Order Determinant

- **Words**   The value of a second-order determinant is found by calculating the difference of the products of the two diagonals.

- **Symbols** $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

Example 1 Second-Order Determinant

Find the value of each determinant.

a. $\begin{vmatrix} -2 & 5 \\ 6 & 8 \end{vmatrix}$

$\begin{vmatrix} -2 & 5 \\ 6 & 8 \end{vmatrix} = (-2)(8) - 5(6)$ Definition of determinant

$= -16 - 30$ Multiply.

$= -46$ Simplify.

b. $\begin{vmatrix} 7 & 4 \\ -3 & 2 \end{vmatrix}$

$\begin{vmatrix} 7 & 4 \\ -3 & 2 \end{vmatrix} = (7)(2) - 4(-3)$ Definition of determinant

$= 14 - (-12)$ Multiply.

$= 26$ Simplify.

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DETERMINANTS OF 3 × 3 MATRICES Determinants of 3×3 matrices are called **third-order determinants**. One method of evaluating third-order determinants is **expansion by minors**. The **minor** of an element is the determinant formed when the row and column containing that element are deleted.

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$ The minor of a_1 is $\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$.

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$ The minor of b_1 is $\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}$.

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$ The minor of c_1 is $\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$.

To use expansion by minors with third-order determinants, each member of one row is multiplied by its minor and its *position sign*, and the results are added together. The position signs alternate between positive and negative, beginning with a positive sign in the first row, first column.

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

Key Concept | *Third–Order Determinant*

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

The definition of third-order determinants shows an expansion using the elements in the first row of the determinant. However, any row can be used.

Example 2 *Expansion by Minors*

Evaluate $\begin{vmatrix} 2 & 7 & -3 \\ -1 & 5 & -4 \\ 6 & 9 & 0 \end{vmatrix}$ **using expansion by minors.**

Decide which row of elements to use for the expansion. For this example, we will use the first row.

$\begin{vmatrix} 2 & 7 & -3 \\ -1 & 5 & -4 \\ 6 & 9 & 0 \end{vmatrix} = 2\begin{vmatrix} 5 & -4 \\ 9 & 0 \end{vmatrix} - 7\begin{vmatrix} -1 & -4 \\ 6 & 0 \end{vmatrix} + (-3)\begin{vmatrix} -1 & 5 \\ 6 & 9 \end{vmatrix}$ Expansion by minors

$= 2(0 - (-36)) - 7(0 - (-24)) - 3(-9 - 30)$ Evaluate 2×2 determinants.

$= 2(36) - 7(24) - 3(-39)$

$= 72 - 168 + 117$ Multiply.

$= 21$ Simplify.

Another method for evaluating a third-order determinant is by using diagonals.

Step 1 Begin by writing the first two columns on the right side of the determinant.

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \implies \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}\begin{matrix} a & b \\ d & e \\ g & h \end{matrix}$$

(continued on the next page)

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Step 2 Next, draw diagonals from each element of the top row of the determinant downward to the right. Find the product of the elements on each diagonal.

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \Rightarrow \begin{matrix} a & b & c & a & b \\ d & e & f & d & e \\ g & h & i & g & h \end{matrix}$$
$$aei \quad bfg \quad cdh$$

Then, draw diagonals from the elements in the third row of the determinant upward to the right. Find the product of the elements on each diagonal.

$$gec \quad hfa \quad idb$$
$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \Rightarrow \begin{matrix} a & b & c & a & b \\ d & e & f & d & e \\ g & h & i & g & h \end{matrix}$$

Step 3 To find the value of the determinant, add the products of the first set of diagonals and then subtract the products of the second set of diagonals. The sum is $aei + bfg + cdh - gec - hfa - idb$.

Example 3 **Use Diagonals**

Evaluate $\begin{vmatrix} -1 & 3 & -3 \\ 4 & -2 & -1 \\ 0 & -5 & 2 \end{vmatrix}$ using diagonals.

Step 1 Rewrite the first two columns to the right of the determinant.

$$\begin{array}{ccc|cc} -1 & 3 & -3 & -1 & 3 \\ 4 & -2 & -1 & 4 & -2 \\ 0 & -5 & 2 & 0 & -5 \end{array}$$

Step 2 Find the products of the elements of the diagonals.

$$\begin{array}{ccc|cc} -1 & 3 & -3 & -1 & 3 \\ 4 & -2 & -1 & 4 & -2 \\ 0 & -5 & 2 & 0 & -5 \end{array} \qquad \begin{matrix} & & 0 & -5 & 24 \\ -1 & 3 & -3 & -1 & 3 \\ 4 & -2 & -1 & 4 & -2 \\ 0 & -5 & 2 & 0 & -5 \end{matrix}$$
$$\qquad\qquad 4 \quad 0 \quad 60$$

Step 3 Add the bottom products and subtract the top products.
$4 + 0 + 60 - 0 - (-5) - 24 = 45$

The value of the determinant is 45.

One very useful application of determinants is finding the areas of polygons. The formula below shows how determinants can be used to find the area of a triangle using the coordinates of the vertices.
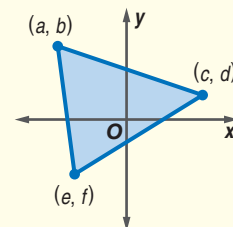
Study Tip

Area Formula
Notice that it is necessary to use the absolute value of A to guarantee a nonnegative value for the area.

Key Concept **Area of a Triangle**

The area of a triangle having vertices at (a, b), (c, d), and (e, f) is $|A|$, where

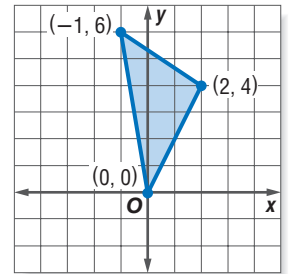$$A = \frac{1}{2}\begin{vmatrix} a & b & 1 \\ c & d & 1 \\ e & f & 1 \end{vmatrix}.$$

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GEOMETRY Find the area of a triangle whose vertices are located at $(-1, 6)$, $(2, 4)$, and $(0, 0)$.

Assign values to a, b, c, d, e, and f and substitute them into the Area Formula. Then evaluate.

$A = \dfrac{1}{2} \begin{vmatrix} a & b & 1 \\ c & d & 1 \\ e & f & 1 \end{vmatrix}$ Area Formula

$= \dfrac{1}{2} \begin{vmatrix} -1 & 6 & 1 \\ 2 & 4 & 1 \\ 0 & 0 & 1 \end{vmatrix}$ $(a, b) = (-1, 6), (c, d) = (2, 4), (e, f) = (0, 0)$

$= \dfrac{1}{2}\left[-1 \begin{vmatrix} 4 & 1 \\ 0 & 1 \end{vmatrix} - 6 \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} + 1 \begin{vmatrix} 2 & 4 \\ 0 & 0 \end{vmatrix} \right]$ Expansion by minors

$= \dfrac{1}{2}[-1(4 - 0) - 6(2 - 0) + 1(0 - 0)]$ Evaluate 2×2 determinants.

$= \dfrac{1}{2}[-4 - 12 + 0]$ Multiply.

$= \dfrac{1}{2}[-16]$ or -8 Simplify.

Remember that the area of a triangle is the absolute value of A. Thus, the area is $|-8|$ or 8 square units.

Check for Understanding

Concept Check

1. **OPEN ENDED** Write a matrix whose determinant is zero.

2. **FIND THE ERROR** Khalid and Erica are finding the determinant of $\begin{bmatrix} 8 & 3 \\ -5 & 2 \end{bmatrix}$.

Khalid	Erica
$\begin{vmatrix} 8 & 3 \\ -5 & 2 \end{vmatrix} = 16 - (-15)$ $= 31$	$\begin{vmatrix} 8 & 3 \\ -5 & 2 \end{vmatrix} = 16 - 15$ $= 1$

Who is correct? Explain your reasoning.

3. **Explain** why $\begin{bmatrix} 2 & 1 & 7 \\ 3 & -5 & 0 \end{bmatrix}$ does not have a determinant.

4. **Find a counterexample** to disprove the following statement.
   Two different matrices can never have the same determinant.

5. **Describe** how to find the minor of 6 in $\begin{bmatrix} 5 & 11 & 7 \\ -1 & 3 & 8 \\ 6 & 0 & -2 \end{bmatrix}$.

6. **Show** that the value of $\begin{vmatrix} -2 & 3 & 5 \\ 0 & -1 & 4 \\ 9 & 7 & 2 \end{vmatrix}$ is the same whether you use expansion by minors or diagonals.

Guided Practice Find the value of each determinant.

7. $\begin{vmatrix} 7 & 8 \\ 3 & -2 \end{vmatrix}$

8. $\begin{vmatrix} -3 & -6 \\ 4 & 8 \end{vmatrix}$

9. $\begin{vmatrix} 0 & 8 \\ 5 & 9 \end{vmatrix}$

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Evaluate each determinant using expansion by minors.

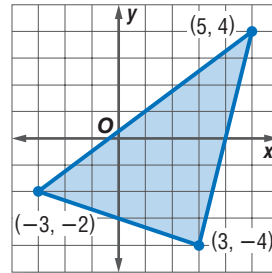10. $\begin{vmatrix} 0 & -4 & 0 \\ 3 & -2 & 5 \\ 2 & -1 & 1 \end{vmatrix}$

11. $\begin{vmatrix} 2 & 3 & 4 \\ 6 & 5 & 7 \\ 1 & 2 & 8 \end{vmatrix}$

Evaluate each determinant using diagonals.

12. $\begin{vmatrix} 1 & 6 & 4 \\ -2 & 3 & 1 \\ 1 & 6 & 4 \end{vmatrix}$

13. $\begin{vmatrix} -1 & 4 & 0 \\ 3 & -2 & -5 \\ -3 & -1 & 2 \end{vmatrix}$

Application   14. **GEOMETRY**   Find the area of the triangle shown at the right.

(5, 4)
(−3, −2)
(3, −4)

Practice and Apply

Find the value of each determinant.

15. $\begin{vmatrix} 10 & 6 \\ 5 & 5 \end{vmatrix}$

16. $\begin{vmatrix} 8 & 5 \\ 6 & 1 \end{vmatrix}$

17. $\begin{vmatrix} -7 & 3 \\ -9 & 7 \end{vmatrix}$

18. $\begin{vmatrix} -2 & 4 \\ 3 & -6 \end{vmatrix}$

19. $\begin{vmatrix} 2 & -7 \\ -5 & 3 \end{vmatrix}$

20. $\begin{vmatrix} -6 & -2 \\ 8 & 5 \end{vmatrix}$

21. $\begin{vmatrix} -9 & 0 \\ -12 & -7 \end{vmatrix}$

22. $\begin{vmatrix} 6 & 14 \\ -3 & -8 \end{vmatrix}$

23. $\begin{vmatrix} 15 & 11 \\ 23 & 19 \end{vmatrix}$

24. $\begin{vmatrix} 21 & 43 \\ 16 & 31 \end{vmatrix}$

25. $\begin{vmatrix} 7 & 5.2 \\ -4 & 1.6 \end{vmatrix}$

26. $\begin{vmatrix} -3.2 & -5.8 \\ 4.1 & 3.9 \end{vmatrix}$

Evaluate each determinant using expansion by minors.

27. $\begin{vmatrix} 3 & 1 & 2 \\ 0 & 6 & 4 \\ 2 & 5 & 1 \end{vmatrix}$

28. $\begin{vmatrix} 7 & 3 & -4 \\ -2 & 9 & 6 \\ 0 & 0 & 0 \end{vmatrix}$

29. $\begin{vmatrix} -2 & 7 & -2 \\ 4 & 5 & 2 \\ 1 & 0 & -1 \end{vmatrix}$

30. $\begin{vmatrix} -3 & 0 & 6 \\ 6 & 5 & -2 \\ 1 & 4 & 2 \end{vmatrix}$

31. $\begin{vmatrix} 1 & 5 & -4 \\ -7 & 3 & 2 \\ 6 & 3 & -1 \end{vmatrix}$

32. $\begin{vmatrix} 3 & 7 & 6 \\ -1 & 6 & 2 \\ 8 & -3 & -5 \end{vmatrix}$

Evaluate each determinant using diagonals.

33. $\begin{vmatrix} 1 & 1 & 1 \\ 3 & 9 & 5 \\ 8 & 7 & 4 \end{vmatrix}$

34. $\begin{vmatrix} 1 & 5 & 2 \\ -6 & -7 & 8 \\ 5 & 9 & -3 \end{vmatrix}$

35. $\begin{vmatrix} 8 & -9 & 0 \\ 1 & 5 & 4 \\ 6 & -2 & 3 \end{vmatrix}$

36. $\begin{vmatrix} 4 & 10 & 7 \\ 3 & 3 & 1 \\ 0 & 5 & 2 \end{vmatrix}$

37. $\begin{vmatrix} 2 & -3 & 4 \\ -2 & 1 & 5 \\ 5 & 3 & -2 \end{vmatrix}$

38. $\begin{vmatrix} 4 & -2 & 3 \\ -2 & 3 & 4 \\ 3 & 4 & 2 \end{vmatrix}$

39. Solve for x if $\det \begin{bmatrix} 2 & x \\ 5 & -3 \end{bmatrix} = 24$.

40. Solve $\det \begin{bmatrix} 4 & x & -2 \\ -x & -3 & 1 \\ -6 & 2 & 3 \end{bmatrix} = -3$ for x.

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41. GEOMETRY Find the area of the polygon shown at the right.
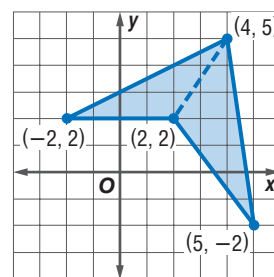
42. GEOMETRY Find the value of x such that the area of a triangle whose vertices have coordinates (6, 5), (8, 2), and $(x, 11)$ is 15 square units.

43. ARCHAEOLOGY During an archaeological dig, a coordinate grid is laid over the site to identify the location of artifacts as they are excavated. During a dig, three corners of a rectangular building have been partially unearthed at $(-1, 6)$, (4, 5), and $(-3, -4)$. If each square on the grid measures one square foot, estimate the area of the floor of the building.

44. GEOGRAPHY Mr. Cardona is a regional sales manager for a company in Florida. Tampa, Orlando, and Ocala outline his region. If a coordinate grid in which 1 unit = 10 miles is placed over the map of Florida with Tampa at the origin, the coordinates of the three cities are (0, 0), (7, 5), and (2.5, 10). Use a determinant to estimate the area of his sales territory.

45. CRITICAL THINKING Find a third-order determinant in which no element is 0, but for which the determinant is 0.

46. CRITICAL THINKING Make a conjecture about how you could find the determinant of a 4×4 matrix using the expansion by minors method. Use a diagram in your explanation.

47. WRITING IN MATH Answer the question that was posed at the beginning of the lesson.

How are determinants used to find areas of polygons?

Include the following in your answer:
- an explanation of how you could use a coordinate grid to estimate the area of the Bermuda Triangle, and
- some advantages of using this method in this situation.
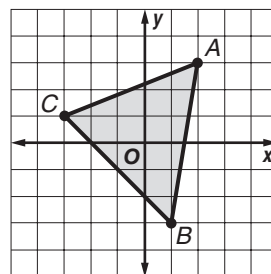
Standardized Test Practice
Ⓐ Ⓑ Ⓒ Ⓓ

48. Find the value of det A if $A = \begin{bmatrix} 0 & 3 & -2 \\ -4 & 0 & 1 \\ 3 & 2 & 0 \end{bmatrix}$.

Ⓐ 0 　　　　 Ⓑ 12 　　　　 Ⓒ 25 　　　　 Ⓓ 36

49. Find the area of triangle ABC.

Ⓐ 10 units^2
Ⓑ 12 units^2
Ⓒ 14 units^2
Ⓓ 16 units^2
Ⓔ none of these

www.algebra2.com/self_check_quiz

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MATRX FUNCTION You can use a TI-83 Plus to find determinants of square matrices using the MATRX functions. Enter the matrix under the NAMES menu. Then use the arrow keys to highlight the MATH menu. Choose det(, which is option 1, to calculate the determinant.

Use a graphing calculator to find the value of each determinant.

50. $\begin{vmatrix} 3 & -6.5 \\ 8 & 3.75 \end{vmatrix}$

51. $\begin{vmatrix} 1.3 & 7.2 \\ 6.1 & 5.4 \end{vmatrix}$

52. $\begin{vmatrix} 6.1 & 4.8 \\ 9.7 & 3.5 \end{vmatrix}$

53. $\begin{vmatrix} 8 & 6 & -5 \\ 10 & -7 & 3 \\ 9 & 14 & -6 \end{vmatrix}$

54. $\begin{vmatrix} 10 & 20 & 30 \\ 40 & 50 & 60 \\ 70 & 80 & 90 \end{vmatrix}$

55. $\begin{vmatrix} 10 & 12 & 4 \\ -3 & 18 & -9 \\ 16 & -2 & -1 \end{vmatrix}$

Maintain Your Skills

Mixed Review

For Exercises 56–58, use the following information.
The vertices of $\triangle ABC$ are $A(-2, 1)$, $B(1, 2)$ and $C(2, -3)$. The triangle is dilated so that its perimeter is $2\frac{1}{2}$ times the original perimeter. *(Lesson 4-4)*

56. Write the coordinates of $\triangle ABC$ in a vertex matrix.

57. Find the coordinates of $\triangle A'B'C'$.

58. Graph $\triangle ABC$ and $\triangle A'B'C'$.

Find each product, if possible. *(Lesson 4-3)*

59. $[5 \quad 2] \cdot \begin{bmatrix} -2 \\ 3 \end{bmatrix}$

60. $\begin{bmatrix} 2 & 4 \\ -2 & 3 \end{bmatrix} \cdot \begin{bmatrix} 3 & 9 \\ -1 & 2 \end{bmatrix}$

61. $\begin{bmatrix} 5 \\ 7 \end{bmatrix} \cdot \begin{bmatrix} 1 & 6 \\ -4 & 2 \end{bmatrix}$

62. $\begin{bmatrix} 7 & 4 \\ -1 & 2 \\ -3 & 5 \end{bmatrix} \cdot [3 \quad 5]$

63. $[4 \quad 2 \quad 0] \cdot \begin{bmatrix} 3 & -2 \\ 1 & 0 \\ 5 & 6 \end{bmatrix}$

64. $\begin{bmatrix} 7 & -5 & 4 \\ 6 & 1 & 3 \end{bmatrix} \cdot \begin{bmatrix} -1 & 3 \\ -2 & -8 \\ 1 & 2 \end{bmatrix}$

65. RUNNING The length of a marathon was determined by the first marathon in the 1908 Olympic Games in London, England. The race began at Windsor Castle and ended in front of the royal box at London's Olympic Stadium, which was a distance of 26 miles 385 yards. Determine how many feet the marathon covers using the formula $f(m, y) = 5280m + 3y$, where m is the number of miles and y is the number of yards. *(Lesson 3-4)*

Write an equation in slope-intercept form for the line that satisfies each set of conditions. *(Lesson 2-4)*

66. slope 1 passes through $(5, 3)$

67. slope $-\frac{4}{3}$ passes through $(6, -8)$

68. passes through $(3, 7)$ and $(-2, -3)$

69. passes through $(0, 5)$ and $(10, 10)$

Getting Ready for the Next Lesson

PREREQUISITE SKILL Solve each system of equations.
*(To review **solving systems of equations**, see Lesson 3-2.)*

70. $x + y = -3$
$3x + 4y = -12$

71. $x + y = 10$
$2x + y = 11$

72. $2x + y = 5$
$4x + y = 9$

73. $3x + 5y = 2$
$2x - y = -3$

74. $6x + 2y = 22$
$3x + 7y = 41$

75. $3x - 2y = -2$
$4x + 7y = 65$

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4-6 Cramer's Rule

Vocabulary
- Cramer's Rule

How is Cramer's Rule used to solve systems of equations?

Two sides of a triangle are contained in lines whose equations are $1.4x + 3.8y = 3.4$ and $2.5x - 1.7y = -10.9$. To find the coordinates of the vertex of the triangle between these two sides, you must solve the system of equations. However, solving this system by using substitution or elimination would require many calculations. Another method for solving systems of equations is Cramer's Rule.

SYSTEMS OF TWO LINEAR EQUATIONS **Cramer's Rule** uses determinants to solve systems of equations. Consider the following system.

$$ax + by = e \quad \text{a, b, c, d, e, and f represent constants, not variables.}$$
$$cx + dy = f$$

Study Tip

Look Back
To review **solving systems of equations**, see Lesson 3-2.

Solve for x by using elimination.

$$adx + bdy = de \qquad \text{Multiply the first equation by } d.$$
$$(-) \, bcx + bdy = bf \qquad \text{Multiply the second equation by } b.$$
$$adx - bcx = de - bf \quad \text{Subtract.}$$
$$(ad - bc)x = de - bf \quad \text{Factor.}$$
$$x = \frac{de - bf}{ad - bc} \qquad \text{Notice that } ad - bc \text{ must not be zero.}$$

Solving for y in the same way produces the following expression.

$$y = \frac{af - ce}{ad - bc}$$

So the solution of the system of equations $ax + by = e$ and $cx + dy = f$ is $\left(\frac{de - bf}{ad - bc}, \frac{af - ce}{ad - bc} \right)$.

Notice that the denominators for each expression are the same. It can be written using a determinant. The numerators can also be written as determinants.

$$ad - bc = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \qquad de - bf = \begin{vmatrix} e & b \\ f & d \end{vmatrix} \qquad af - ce = \begin{vmatrix} a & e \\ c & f \end{vmatrix}$$

Key Concept   *Cramer's Rule for Two Variables*

The solution of the system of linear equations
$$ax + by = e$$
$$cx + dy = f$$

is (x, y), where $x = \dfrac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$, $y = \dfrac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$, and $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$.

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Example 1 **System of Two Equations**

Use Cramer's Rule to solve the system of equations.

$5x + 7y = 13$
$2x - 5y = 13$

$$x = \dfrac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} \qquad \text{Cramer's Rule} \qquad y = \dfrac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$$

$$= \dfrac{\begin{vmatrix} 13 & 7 \\ 13 & -5 \end{vmatrix}}{\begin{vmatrix} 5 & 7 \\ 2 & -5 \end{vmatrix}} \qquad \begin{array}{c} a = 5, b = 7, c = 2, d = -5, \\ e = 13, \text{ and } f = 13 \end{array} \qquad = \dfrac{\begin{vmatrix} 5 & 13 \\ 2 & 13 \end{vmatrix}}{\begin{vmatrix} 5 & 7 \\ 2 & -5 \end{vmatrix}}$$

$$= \dfrac{13(-5) - 13(7)}{5(-5) - 2(7)} \qquad \text{Evaluate each determinant.} \qquad = \dfrac{5(13) - 2(13)}{5(-5) - 2(7)}$$

$$= \dfrac{-156}{-39} \text{ or } 4 \qquad \text{Simplify.} \qquad = \dfrac{39}{-39} \text{ or } -1$$

The solution is $(4, -1)$.

Cramer's Rule is especially useful when the coefficients are large or involve fractions or decimals.

Example 2 **Use Cramer's Rule**

Elections •··········

ELECTIONS In the 2000 presidential election, George W. Bush received about 8,400,000 votes in California and Texas while Al Gore received about 8,300,000 votes in those states. The graph shows the percent of the popular vote that each candidate received in those states.

a. Write a system of equations that represents the total number of votes cast for each candidate in these two states.

Let x represent the total number of votes in California.

Let y represent the total number of votes in Texas.

$0.42x + 0.59y = 8,400,000$ Votes for Bush

$0.53x + 0.38y = 8,300,000$ Votes for Gore

Presidential Elections

Source: States' Elections Offices

b. Find the total number of popular votes cast in California and in Texas.

$$x = \dfrac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} \qquad \text{Cramer's Rule} \qquad y = \dfrac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$$
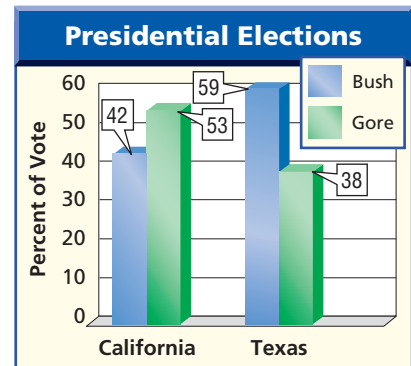
$$= \dfrac{\begin{vmatrix} 8,400,000 & 0.59 \\ 8,300,000 & 0.38 \end{vmatrix}}{\begin{vmatrix} 0.42 & 0.59 \\ 0.53 & 0.38 \end{vmatrix}} \qquad\qquad = \dfrac{\begin{vmatrix} 0.42 & 8,400,000 \\ 0.53 & 8,300,000 \end{vmatrix}}{\begin{vmatrix} 0.42 & 0.59 \\ 0.53 & 0.38 \end{vmatrix}}$$

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$$= \frac{8,400,000(0.38) - 8,300,000(0.59)}{0.42(0.38) - 0.53(0.59)} \qquad = \frac{0.42(8,300,000) - 0.53(8,400,000)}{0.42(0.38) - 0.53(0.59)}$$

$$= \frac{-1,705,000}{-0.1531} \qquad\qquad\qquad = \frac{-966,000}{-0.1531}$$

$$\approx 11,136,512.08 \qquad\qquad\qquad \approx 6,309,601.57$$

The solution of the system is about (11,136,512.08, 6,309,601.57).

So, there were about 11,100,000 popular votes cast in California and about 6,300,000 popular votes cast in Texas.

SYSTEMS OF THREE LINEAR EQUATIONS
You can also use Cramer's Rule to solve a system of three equations in three variables.

Key Concept — Cramer's Rule for Three Variables

The solution of the system whose equations are
$$ax + by + cz = j$$
$$dx + ey + fz = k$$
$$gx + hy + iz = \ell$$

is (x, y, z), where $x = \dfrac{\begin{vmatrix} j & b & c \\ k & e & f \\ \ell & h & i \end{vmatrix}}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}}$, $y = \dfrac{\begin{vmatrix} a & j & c \\ d & k & f \\ g & \ell & i \end{vmatrix}}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}}$, $z = \dfrac{\begin{vmatrix} a & b & j \\ d & e & k \\ g & h & \ell \end{vmatrix}}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}}$, and $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \neq 0$.

Example 3 System of Three Equations

Use Cramer's Rule to solve the system of equations.

$$3x + y + z = -1$$
$$-6x + 5y + 3z = -9$$
$$9x - 2y - z = 5$$

$$x = \frac{\begin{vmatrix} j & b & c \\ k & e & f \\ \ell & h & i \end{vmatrix}}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}} \qquad y = \frac{\begin{vmatrix} a & j & c \\ d & k & f \\ g & \ell & i \end{vmatrix}}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}} \qquad z = \frac{\begin{vmatrix} a & b & j \\ d & e & k \\ g & h & \ell \end{vmatrix}}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}}$$

$$= \frac{\begin{vmatrix} -1 & 1 & 1 \\ -9 & 5 & 3 \\ 5 & -2 & -1 \end{vmatrix}}{\begin{vmatrix} 3 & 1 & 1 \\ -6 & 5 & 3 \\ 9 & -2 & -1 \end{vmatrix}} \qquad = \frac{\begin{vmatrix} 3 & -1 & 1 \\ -6 & -9 & 3 \\ 9 & 5 & -1 \end{vmatrix}}{\begin{vmatrix} 3 & 1 & 1 \\ -6 & 5 & 3 \\ 9 & -2 & -1 \end{vmatrix}} \qquad = \frac{\begin{vmatrix} 3 & 1 & -1 \\ -6 & 5 & -9 \\ 9 & -2 & 5 \end{vmatrix}}{\begin{vmatrix} 3 & 1 & 1 \\ -6 & 5 & 3 \\ 9 & -2 & -1 \end{vmatrix}}$$

Use a calculator to evaluate each determinant.

$$x = \frac{-2}{-9} \text{ or } \frac{2}{9} \qquad y = \frac{12}{-9} \text{ or } -\frac{4}{3} \qquad z = \frac{3}{-9} \text{ or } -\frac{1}{3}$$

The solution is $\left(\dfrac{2}{9}, -\dfrac{4}{3}, -\dfrac{1}{3}\right)$.

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Check for Understanding

Concept Check
1. **Describe** the condition that must be met in order to use Cramer's Rule.
2. **OPEN ENDED** Write a system of equations that *cannot* be solved using Cramer's Rule.
3. **Write** a system of equations whose solution is $x = \dfrac{\begin{vmatrix} -6 & 5 \\ 30 & -2 \end{vmatrix}}{\begin{vmatrix} 3 & 5 \\ 4 & -2 \end{vmatrix}}$, $y = \dfrac{\begin{vmatrix} 3 & -6 \\ 4 & 30 \end{vmatrix}}{\begin{vmatrix} 3 & 5 \\ 4 & -2 \end{vmatrix}}$.

Guided Practice Use Cramer's Rule to solve each system of equations.

4. $x - 4y = 1$
   $2x + 3y = 13$

5. $0.2a = 0.3b$
   $0.4a - 0.2b = 0.2$

6. $\frac{1}{2}r - \frac{2}{3}s = 2\frac{1}{3}$
   $\frac{3}{5}r + \frac{4}{5}s = -10$

7. $2x - y + 3z = 5$
   $3x + 2y - 5z = 4$
   $x - 4y + 11z = 3$

8. $a + 9b - 2c = 2$
   $-a - 3b + 4c = 1$
   $2a + 3b - 6c = -5$

9. $r + 4s + 3t = 10$
   $2r - 2s + t = 15$
   $r + 2s - 3t = -1$

Application **INVESTING** For Exercises 10 and 11, use the following information.
Jarrod Wright has $4000 he would like to invest so that he can earn some interest on it. He has discovered that he could put it in a savings account paying 6.5% interest annually, or in a certificate of deposit with an annual rate of 8%. He wants his interest for the year to be $297.50, because earning more than this would put him into a higher tax bracket.

10. Write a system of equations, in which the unknowns s and d stand for the amounts of money Jarrod should deposit in the savings account and the certificate of deposit, respectively.

11. How much should he put in a savings account, and how much should he put in the certificate of deposit?

Practice and Apply

Use Cramer's Rule to solve each system of equations.

12. $5x + 2y = 8$
    $2x - 3y = 7$

13. $2m + 7n = 4$
    $m - 2n = -20$

14. $2r - s = 1$
    $3r + 2s = 19$

15. $3a + 5b = 33$
    $5a + 7b = 51$

16. $2m - 4n = -1$
    $3n - 4m = -5$

17. $4x + 3y = 6$
    $8x - y = -9$

18. $0.5r - s = -1$
    $0.75r + 0.5s = -0.25$

19. $1.5m - 0.7n = 0.5$
    $2.2m - 0.6n = -7.4$

20. $3x - 2y = 4$
    $\frac{1}{2}x - \frac{2}{3}y = 1$

21. $2a + 3b = -16$
    $\frac{3}{4}a - \frac{7}{8}b = 10$

22. $\frac{1}{3}r + \frac{2}{5}s = 5$
    $\frac{2}{3}r - \frac{1}{2}s = -3$

23. $\frac{3}{4}x + \frac{1}{2}y = \frac{11}{12}$
    $\frac{1}{2}x - \frac{1}{4}y = \frac{1}{8}$

24. **GEOMETRY** The two sides of an angle are contained in lines whose equations are $4x + y = -4$ and $2x - 3y = -9$. Find the coordinates of the vertex of the angle.

25. **GEOMETRY** Two sides of a parallelogram are contained in the lines whose equations are $2.3x + 1.2y = 2.1$ and $4.1x - 0.5y = 14.3$. Find the coordinates of a vertex of the parallelogram.

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Use Cramer's Rule to solve each system of equations.

26. $x + y + z = 6$
$2x + y - 4z = -15$
$5x - 3y + z = -10$

27. $a - 2b + c = 7$
$6a + 2b - 2c = 4$
$4a + 6b + 4c = 14$

28. $r - 2s - 5t = -1$
$r + 2s - 2t = 5$
$4r + s + t = -1$

29. $3a + c = 23$
$4a + 7b - 2c = -22$
$8a - b - c = 34$

30. $4x + 2y - 3z = -32$
$-x - 3y + z = 54$
$2y + 8z = 78$

31. $2r + 25s = 40$
$10r + 12s + 6t = -2$
$36r - 25s + 50t = -10$

GAMES For Exercises 32 and 33, use the following information.
Marcus purchased a game card to play virtual games at the arcade. His favorite games are the race car simulator, which costs 7 points for each play, and the snowboard simulator, which costs 5 points for each play. Marcus came with enough money to buy a 50-point card, and he has time to play 8 games.

32. Write a system of equations.

33. Solve the system using Cramer's Rule to find the number of times Marcus can play race car simulator and snowboard simulator.

INTERIOR DESIGN For Exercises 34 and 35, use the following information.
An interior designer is preparing invoices for two of her clients. She has ordered silk dupioni and cotton damask fabric for both of them.

Client	Fabric	Yards	Total Cost
Harada	silk	8	$604.79
	cotton	13	
Martina	silk	$5\frac{1}{2}$	$542.30
	cotton	14	

34. Write a system of two equations using the information given.

35. Find the price per yard of each fabric.

PRICING For Exercises 36 and 37, use the following information.
The Harvest Nut Company sells made-to-order trail mixes. Santito's favorite mix contains peanuts, raisins, and carob-coated pretzels. Peanuts sell for $3.20 per pound, raisins are $2.40 per pound, and the carob-coated pretzels are $4.00 per pound. Santito chooses to have twice as many pounds of pretzels as raisins, wants 5 pounds of mix, and can afford $16.80.

36. Write a system of three equations using the information given.

37. How many pounds of peanuts, raisins, and carob-coated pretzels can Santito buy?

38. CRITICAL THINKING In Cramer's Rule, if the value of the determinant is zero, what must be true of the graph of the system of equations represented by the determinant? Give examples to support your answer.

39. WRITING IN MATH   Answer the question that was posed at the beginning of the lesson.

How is Cramer's Rule used to solve systems of equations?

Include the following in your answer.
- an explanation of how Cramer's rule uses determinants, and
- a situation where Cramer's rule would be easier to solve a system of equations than substitution or elimination and why.

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40. Use Cramer's Rule to solve the system of equations $3x + 8y = 28$ and $5x - 7y = -55$.

 Ⓐ $(3, 5)$ Ⓑ $(-4, 5)$ Ⓒ $(4, 2)$ Ⓓ $(-5, 4)$

41. SHORT RESPONSE Find the measures of $\angle ABC$ and $\angle CBD$.
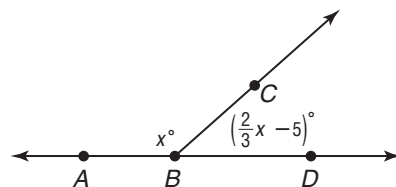
Maintain Your Skills

Mixed Review

Find the value of each determinant. *(Lesson 4-5)*

42. $\begin{vmatrix} 3 & 2 \\ -2 & 4 \end{vmatrix}$

43. $\begin{vmatrix} 8 & 6 \\ 4 & 8 \end{vmatrix}$

44. $\begin{vmatrix} -5 & 2 \\ 4 & 9 \end{vmatrix}$

For Exercises 45–47, use the following information.
Triangle ABC with vertices $A(0, 2)$, $B(-3, -1)$, and $C(-2, -4)$ is translated 1 unit right and 3 units up. *(Lesson 4-4)*

45. Write the translation matrix.

46. Find the coordinates of $\triangle A'B'C'$.

47. Graph the preimage and the image.

Solve each system of equations by graphing. *(Lesson 3-1)*

48. $y = 3x + 5$
$y = -2x - 5$

49. $x + y = 7$
$\frac{1}{2}x - y = -1$

50. $x - 2y = 10$
$2x - 4y = 12$

51. BUSINESS The Friendly Fix-It Company charges a base fee of \$35 for any in-home repair. In addition, the technician charges \$10 per hour. Write an equation for the cost c of an in-home repair of h hours. *(Lesson 1-3)*

Getting Ready for the Next Lesson

PREREQUISITE SKILL Find each product, if possible.
*(To review **multiplying matrices**, see Lesson 4-3.)*

52. $[2 \quad 5] \cdot \begin{bmatrix} 3 & 1 \\ -2 & 6 \end{bmatrix}$

53. $\begin{bmatrix} 0 & 9 \\ 5 & 7 \end{bmatrix} \cdot \begin{bmatrix} 2 & -6 \\ 8 & 1 \end{bmatrix}$

54. $\begin{bmatrix} 5 & -4 \\ 8 & 3 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 1 \end{bmatrix}$

Practice Quiz 2 — Lessons 4-4 through 4-6

For Exercises 1–3, reflect square $ABCD$ with vertices $A(1, 2)$, $B(4, -1)$, $C(1, -4)$, and $D(-2, -1)$ over the y-axis. *(Lesson 4-4)*

1. Write the coordinates in a vertex matrix.

2. Find the coordinates of $A'B'C'D'$.

3. Graph $ABCD$ and $A'B'C'D'$.

Find the value of each determinant. *(Lesson 4-5)*

4. $\begin{vmatrix} 3 & -2 \\ 5 & 4 \end{vmatrix}$

5. $\begin{vmatrix} -8 & 3 \\ 6 & 5 \end{vmatrix}$

6. $\begin{vmatrix} 1 & 3 & -2 \\ 7 & 0 & 4 \\ -3 & 5 & -1 \end{vmatrix}$

7. $\begin{vmatrix} 3 & 4 & 4 \\ 2 & 1 & 5 \\ 0 & -8 & 6 \end{vmatrix}$

Use Cramer's Rule to solve each system of equations. *(Lesson 4-6)*

8. $3x - 2y = 7$
$4x - y = 6$

9. $7r + 5s = 3$
$3r - 2s = 22$

10. $3a - 5b + 2c = -5$
$4a + b + 3c = 9$
$2a - c = 1$

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Identity and Inverse Matrices

What You'll Learn

- Determine whether two matrices are inverses.
- Find the inverse of a 2×2 matrix.

Vocabulary

- identity matrix
- inverse

How are inverse matrices used in cryptography?

With the rise of Internet shopping, ensuring the privacy of the user's personal information has become an important priority. Companies protect their computers by using codes. Cryptography is a method of preparing coded messages that can only be deciphered by using the "key" to the message.
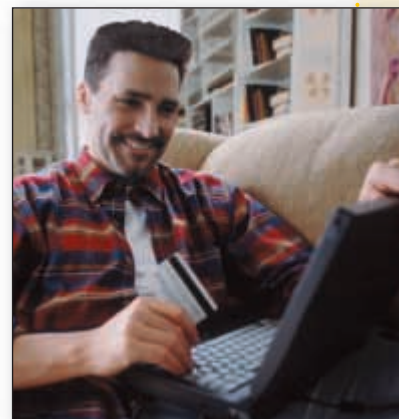
The following technique is a simplified version of how cryptography works.

- First, assign a number to each letter of the alphabet.
- Convert your message into a matrix and multiply it by the coding matrix. The message is now unreadable to anyone who does not have the key to the code.
- To decode the message, the recipient of the coded message would multiply by the opposite, or inverse, of the coding matrix.

Code								
_ 0	**A** 1	**B** 2	**C** 3	**D** 4	**E** 5	**F** 6	**G** 7	**H** 8
I 9	**J** 10	**K** 11	**L** 12	**M** 13	**N** 14	**O** 15	**P** 16	**Q** 17
R 18	**S** 19	**T** 20	**U** 21	**V** 22	**W** 23	**X** 24	**Y** 25	**Z** 26

IDENTITY AND INVERSE MATRICES   Recall that in real numbers, two numbers are inverses if their product is the identity, 1. Similarly, for matrices, the identity matrix is a square matrix that, when multiplied by another matrix, equals that same matrix. If A is any $n \times n$ matrix and I is the $n \times n$ identity matrix, then $A \cdot I = A$ and $I \cdot A = A$.

2 × 2 Identity Matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

3 × 3 Identity Matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Key Concept — Identity Matrix for Multiplication

- **Words** The identity matrix for multiplication I is a square matrix with 1 for every element of the main diagonal, from upper left to lower right, and 0 in all other positions. For any square matrix A of the same dimension as I, $A \cdot I = I \cdot A = A$.

- **Symbols** If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ such that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Two $n \times n$ matrices are **inverses** of each other if their product is the identity matrix. If matrix A has an inverse symbolized by A^{-1}, then $A \cdot A^{-1} = A^{-1} \cdot A = I$.

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Example 1 *Verify Inverse Matrices*

Determine whether each pair of matrices are inverses.

a. $X = \begin{bmatrix} 2 & 2 \\ -1 & 4 \end{bmatrix}$ and $Y = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -1 & \frac{1}{4} \end{bmatrix}$

Check to see if $X \cdot Y = I$.

$X \cdot Y = \begin{bmatrix} 2 & 2 \\ -1 & 4 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -1 & \frac{1}{4} \end{bmatrix}$ — Write an equation.

$= \begin{bmatrix} 1-2 & 1+\frac{1}{2} \\ -\frac{1}{2}+(-4) & -\frac{1}{2}+1 \end{bmatrix}$ or $\begin{bmatrix} -1 & 1\frac{1}{2} \\ -4\frac{1}{2} & \frac{1}{2} \end{bmatrix}$ — Matrix multiplication

Since $X \cdot Y \neq I$, they are *not* inverses.

b. $P = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$ and $Q = \begin{bmatrix} 1 & -2 \\ -\frac{1}{2} & \frac{3}{2} \end{bmatrix}$

Find $P \cdot Q$.

$P \cdot Q = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & -2 \\ -\frac{1}{2} & \frac{3}{2} \end{bmatrix}$ — Write an equation.

$= \begin{bmatrix} 3-2 & -6+6 \\ 1-1 & -2+3 \end{bmatrix}$ or $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ — Matrix multiplication

Now find $Q \cdot P$.

$Q \cdot P = \begin{bmatrix} 1 & -2 \\ -\frac{1}{2} & \frac{3}{2} \end{bmatrix} \cdot \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$ — Write an equation.

$= \begin{bmatrix} 3-2 & 4-4 \\ -\frac{3}{2}+\frac{3}{2} & -2+3 \end{bmatrix}$ or $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ — Matrix multiplication

Since $P \cdot Q = Q \cdot P = I$, P and Q are inverses.

FIND INVERSE MATRICES Some matrices do not have an inverse. You can determine whether a matrix has an inverse by using the determinant.

Key Concept *Inverse of a 2 × 2 Matrix*

The inverse of matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $A^{-1} = \dfrac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, where $ad - bc \neq 0$.

Notice that $ad - bc$ is the value of det A. Therefore, if the value of the determinant of a matrix is 0, the matrix cannot have an inverse.

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Example 2 · Find the Inverse of a Matrix

Find the inverse of each matrix, if it exists.

a. $R = \begin{bmatrix} -4 & -3 \\ 8 & 6 \end{bmatrix}$

Find the value of the determinant.

$\begin{vmatrix} -4 & -3 \\ 8 & 6 \end{vmatrix} = -24 - (-24) = 0$

Since the determinant equals 0, R^{-1} does not exist.

b. $P = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$

Find the value of the determinant.

$\begin{vmatrix} 3 & 1 \\ 5 & 2 \end{vmatrix} = 6 - 5 \text{ or } 1$

Since the determinant does not equal 0, P^{-1} exists.

$P^{-1} = \dfrac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ **Definition of inverse**

$\quad = \dfrac{1}{3(2) - 1(5)} \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}$ **$a = 3, b = 1, c = 5, d = 1$**

$\quad = 1 \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix} \text{ or } \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}$ **Simplify.**

CHECK $\begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix} \cdot \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix} = \begin{bmatrix} 6 - 5 & 2 - 2 \\ -15 + 15 & -5 + 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ✓

Matrices can be used to code messages by placing the message in a $2 \times n$ matrix.

Example 3 · Use Inverses to Solve a Problem

a. **CRYPTOGRAPHY** Use the table at the beginning of the lesson to assign a number to each letter in the message GO_TONIGHT. Then code the message with the matrix $A = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}$.

Convert the message to numbers using the table.

G O _ T O N I G H T
7 | 15 | 0 | 20 | 15 | 14 | 9 | 7 | 8 | 20

Write the message in matrix form. Then multiply the message matrix B by the coding matrix A.

$BA = \begin{bmatrix} 7 & 15 \\ 0 & 20 \\ 15 & 14 \\ 9 & 7 \\ 8 & 20 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}$ **Write an equation.**

$= \begin{bmatrix} 14 + 60 & 7 + 45 \\ 0 + 80 & 0 + 60 \\ 30 + 56 & 15 + 42 \\ 18 + 28 & 9 + 21 \\ 16 + 80 & 8 + 60 \end{bmatrix}$ **Matrix multiplication**

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Study Tip

Messages
If there is an odd number
of letters to be coded, add
a 0 at the end of the
message.

$$= \begin{bmatrix} 74 & 52 \\ 80 & 60 \\ 86 & 57 \\ 46 & 30 \\ 96 & 68 \end{bmatrix} \qquad \text{Simplify.}$$

The coded message is $74 \mid 52 \mid 80 \mid 60 \mid 86 \mid 57 \mid 46 \mid 30 \mid 96 \mid 68$.

b. Use the inverse matrix A^{-1} to decode the message in Example 3a.

First find the inverse matrix of $A = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}$.

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \qquad \text{Definition of inverse}$$

$$= \frac{1}{2(3) - (1)(4)} \begin{bmatrix} 3 & -1 \\ -4 & 2 \end{bmatrix} \qquad a = 2, b = 1, c = 4, d = 3$$

$$= \frac{1}{2} \begin{bmatrix} 3 & -1 \\ -4 & 2 \end{bmatrix} \text{ or } \begin{bmatrix} \frac{3}{2} & -\frac{1}{2} \\ -2 & 1 \end{bmatrix} \qquad \text{Simplify.}$$

Next, decode the message by multiplying the coded matrix C by A^{-1}.

$$CA^{-1} = \begin{bmatrix} 74 & 52 \\ 80 & 60 \\ 86 & 57 \\ 46 & 30 \\ 96 & 68 \end{bmatrix} \cdot \begin{bmatrix} \frac{3}{2} & -\frac{1}{2} \\ -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 111 - 104 & -37 + 52 \\ 120 - 120 & -40 + 60 \\ 129 - 114 & -43 + 57 \\ 69 - 60 & -23 + 30 \\ 144 - 136 & -48 + 68 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 15 \\ 0 & 20 \\ 15 & 14 \\ 9 & 7 \\ 8 & 20 \end{bmatrix}$$

Use the table again to convert the numbers to letters. You can now read the message.

$7 \mid 15 \mid 0 \mid 20 \mid 15 \mid 14 \mid 9 \mid 7 \mid 8 \mid 20$

G O _ T O N I G H T

Check for Understanding

Concept Check

1. **Write** the 4×4 identity matrix.

2. **Explain** how to find the inverse of a 2×2 matrix.

3. **OPEN ENDED** Create a square matrix that does not have an inverse.

CONTENTS

Guided Practice Determine whether each pair of matrices are inverses.

4. $A = \begin{bmatrix} 2 & -1 \\ 1 & -3 \end{bmatrix}$, $B = \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{3} \\ 0 & -\frac{1}{3} \end{bmatrix}$

5. $X = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$, $Y = \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}$

Find the inverse of each matrix, if it exists.

6. $\begin{bmatrix} 8 & -5 \\ -3 & 2 \end{bmatrix}$

7. $\begin{bmatrix} 4 & -8 \\ -1 & 2 \end{bmatrix}$

8. $\begin{bmatrix} -5 & 1 \\ 7 & 4 \end{bmatrix}$

Application

9. **CRYPTOGRAPHY** Select a headline from a newspaper or the title of a magazine article and code it using your own coding matrix. Give your message and the coding matrix to a friend to decode. (*Hint*: Use a coding matrix whose determinant is 1 and that has all positive elements.)

Practice and Apply

Determine whether each pair of matrices are inverses.

10. $P = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$, $Q = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$

11. $R = \begin{bmatrix} 2 & 2 \\ 3 & 4 \end{bmatrix}$, $S = \begin{bmatrix} 2 & -1 \\ -\frac{3}{2} & 1 \end{bmatrix}$

12. $A = \begin{bmatrix} 6 & 2 \\ 5 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ -\frac{5}{2} & -3 \end{bmatrix}$

13. $X = \begin{bmatrix} \frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{bmatrix}$, $Y = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

14. $C = \begin{bmatrix} 1 & 5 \\ 1 & -2 \end{bmatrix}$, $D = \begin{bmatrix} \frac{2}{7} & \frac{5}{7} \\ \frac{1}{7} & -\frac{1}{7} \end{bmatrix}$

15. $J = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 1 & 1 & 2 \end{bmatrix}$, $K = \begin{bmatrix} -\frac{5}{4} & \frac{1}{4} & \frac{7}{4} \\ \frac{3}{4} & \frac{1}{4} & -\frac{5}{4} \\ \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} \end{bmatrix}$

Determine whether each statement is *true* or *false*.

16. Only square matrices have multiplicative identities.

17. Only square matrices have multiplicative inverses.

18. Some square matrices do not have multiplicative inverses.

19. Some square matrices do not have multiplicative identities.

Find the inverse of each matrix, if it exists.

20. $\begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}$

21. $\begin{bmatrix} 6 & 3 \\ 8 & 4 \end{bmatrix}$

22. $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

23. $\begin{bmatrix} 3 & 1 \\ -4 & 1 \end{bmatrix}$

24. $\begin{bmatrix} -3 & -2 \\ 6 & 4 \end{bmatrix}$

25. $\begin{bmatrix} -3 & 7 \\ 2 & -6 \end{bmatrix}$

26. $\begin{bmatrix} 4 & -3 \\ 2 & 7 \end{bmatrix}$

27. $\begin{bmatrix} -2 & 0 \\ 5 & 6 \end{bmatrix}$

28. $\begin{bmatrix} -4 & 6 \\ 6 & -9 \end{bmatrix}$

29. $\begin{bmatrix} 2 & -5 \\ 6 & 1 \end{bmatrix}$

30. $\begin{bmatrix} \frac{1}{2} & -\frac{3}{4} \\ \frac{1}{6} & \frac{1}{4} \end{bmatrix}$

31. $\begin{bmatrix} \frac{3}{10} & \frac{5}{8} \\ \frac{1}{5} & \frac{3}{4} \end{bmatrix}$

www.algebra2.com/self_check_quiz

CONTENTS

32. Compare the matrix used to reflect a figure over the x-axis to the matrix used to reflect a figure over the y-axis.

a. Are they inverses?

b. Does your answer make sense based on the geometry? Use a drawing to support your answer.

33. The matrix used to rotate a figure $270°$ counterclockwise about the origin is $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Compare this matrix with the matrix used to rotate a figure $90°$ counterclockwise about the origin.

a. Are they inverses?

b. Does your answer make sense based on the geometry? Use a drawing to support your answer.

GEOMETRY For Exercises 34–38, use the figure below.

34. Write the vertex matrix A for the rectangle.

35. Use matrix multiplication to find BA if $B = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$.

36. Graph the vertices of the transformed rectangle. Describe the transformation.

37. Make a conjecture about what transformation B^{-1} describes on a coordinate plane.

38. Test your conjecture. Find B^{-1} and multiply it by the result of BA. Make a drawing to verify your conjecture.
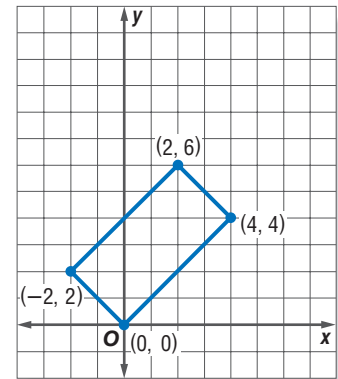
CRYPTOGRAPHY For Exercises 39–41, use the alphabet table below.

Your friend has sent you a series of messages that were coded with the coding matrix $C = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$. Use the inverse of matrix C to decode each message.

39. $50|36|51|29|18|18|26|13|33|26|44|$ $22|48|33|59|34|61|35|4|2$

40. $59|33|8|8|39|21|7|7|56|37|25|16|$ $4|2$

41. $59|34|49|31|40|20|16|14|21|15|25|$ $25|36|24|32|16$

Code		
A 26	J 17	S 8
B 25	K 16	T 7
C 24	L 15	U 6
D 23	M 14	V 5
E 22	N 13	W 4
F 21	O 12	X 3
G 20	P 11	Y 2
H 19	Q 10	Z 1
I 18	R 9	_ 0

42. RESEARCH Use the Internet or other reference to find examples of codes used throughout history. Explain how messages were coded.

43. CRITICAL THINKING For which values of a, b, c, and d will $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = A^{-1}$?

44. WRITING IN MATH Answer the question that was posed at the beginning of the lesson.

How are inverse matrices used in cryptography?

Include the following in your answer:

- an explanation of why the inverse matrix works in decoding a message, and
- a description of the conditions you must consider when writing a message in matrix form.

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45. What is the inverse of $\begin{bmatrix} 4 & 1 \\ 10 & 2 \end{bmatrix}$?

Ⓐ $\begin{bmatrix} -1 & \frac{1}{2} \\ 5 & -2 \end{bmatrix}$ Ⓑ $\begin{bmatrix} 2 & -1 \\ -10 & 4 \end{bmatrix}$ Ⓒ $\begin{bmatrix} 1 & 5 \\ \frac{1}{2} & 2 \end{bmatrix}$ Ⓓ $\begin{bmatrix} -2 & \frac{1}{2} \\ 5 & -1 \end{bmatrix}$

46. Which matrix does *not* have an inverse?

Ⓐ $\begin{bmatrix} 5 & 3 \\ 2 & 4 \end{bmatrix}$ Ⓑ $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ Ⓒ $\begin{bmatrix} -3 & -3 \\ 6 & -6 \end{bmatrix}$ Ⓓ $\begin{bmatrix} -10 & -5 \\ 8 & 4 \end{bmatrix}$

Graphing Calculator

INVERSE FUNCTION   The $\boxed{x^{-1}}$ key on a TI-83 Plus is used to find the inverse of a matrix. If you get a **SINGULAR MATRIX** error on the screen, then the matrix has no inverse.

Use a graphing calculator to find the inverse of each matrix.

47. $\begin{bmatrix} -11 & 9 \\ 6 & -5 \end{bmatrix}$

48. $\begin{bmatrix} 12 & 4 \\ 15 & 5 \end{bmatrix}$

49. $\begin{bmatrix} 2 & -1 \\ 1 & -3 \end{bmatrix}$

50. $\begin{bmatrix} 25 & -4 \\ -35 & 6 \end{bmatrix}$

51. $\begin{bmatrix} 2 & 5 & 2 \\ 1 & 4 & 1 \\ 6 & 3 & 3 \end{bmatrix}$

52. $\begin{bmatrix} 3 & 1 & 2 \\ -2 & 0 & 4 \\ 3 & 5 & 2 \end{bmatrix}$

Maintain Your Skills

Mixed Review

Use Cramer's Rule to solve each system of equations.   *(Lesson 4-6)*

53. $3x + 2y = -2$
$x - 3y = 14$

54. $2x + 5y = 35$
$7x - 4y = -28$

55. $4x - 3z = -23$
$-2x - 5y + z = -9$
$y - z = 3$

Evaluate each determinant by using diagonals or expansion by minors.   *(Lesson 4-5)*

56. $\begin{vmatrix} 2 & 8 & -6 \\ 4 & 5 & 2 \\ -3 & -6 & -1 \end{vmatrix}$

57. $\begin{vmatrix} -3 & -3 & 1 \\ -9 & -2 & 3 \\ 5 & -2 & -1 \end{vmatrix}$

58. $\begin{vmatrix} 5 & -7 & 3 \\ -1 & 2 & -9 \\ 5 & -7 & 3 \end{vmatrix}$

Find the slope of the line that passes through each pair of points.   *(Lesson 2-3)*

59. $(2, 5), (6, 9)$
60. $(1, 0), (-2, 9)$
61. $(-5, 4), (-3, -6)$
62. $(-2, 2), (-5, 1)$
63. $(0, 3), (-2, -2)$
64. $(-8, 9), (0, 6)$

65. OCEANOGRAPHY   The deepest point in any ocean, the bottom of the Mariana Trench in the Pacific Ocean, is 6.8 miles below sea level. Water pressure in the ocean is represented by the function $f(x) = 1.15x$, where x is the depth in miles and $f(x)$ is the pressure in tons per square inch. Find the water pressure at the deepest point in the Mariana Trench.   *(Lesson 2-1)*

Evaluate each expression.   *(Lesson 1-1)*

66. $3(2^3 + 1)$
67. $7 - 5 \div 2 + 1$
68. $\dfrac{9 - 4 \cdot 3}{6}$
69. $[40 - (7 + 9)] \div 8$
70. $[(-2 + 8)6 + 1]8$
71. $(4 - 1)(8 + 2)^2$

Getting Ready for the Next Lesson

PREREQUISITE SKILL   Solve each equation.
*(To review **solving multi-step equations**, see Lesson 1-3.)*

72. $3k + 8 = 5$
73. $12 = -5h + 2$
74. $7z - 4 = 5z + 8$
75. $\dfrac{x}{2} + 5 = 7$
76. $\dfrac{3 + n}{6} = -4$
77. $6 = \dfrac{s - 8}{-7}$

CONTENTS

Using Matrices to Solve Systems of Equations

What You'll Learn

- Write matrix equations for systems of equations.
- Solve systems of equations using matrix equations.

Vocabulary

- matrix equation

How can matrices be used in population ecology?

Population ecology is the study of a species or a group of species that inhabits the same area. A biologist is studying two species of birds that compete for food and territory. He estimates that a particular region with an area of 14.25 acres (approximately 69,000 square yards) can supply 20,000 pounds of food for the birds during their nesting season. Species A needs 140 pounds of food and has a territory of 500 square yards per nesting pair. Species B needs 120 pounds of food and has a territory of 400 square yards per nesting pair. The biologist can use this information to find the number of birds of each species that the area can support.

WRITE MATRIX EQUATIONS The situation above can be represented using a system of equations that can be solved using matrices. Consider the system of equations below. You can write this system with matrices by using the left and right sides of the equations.

$$5x + 7y = 11 \quad \rightarrow \quad \begin{bmatrix} 5x + 7y \\ 3x + 8y \end{bmatrix} = \begin{bmatrix} 11 \\ 18 \end{bmatrix}$$
$$3x + 8y = 18$$

Write the matrix on the left as the product of the coefficients and the variables.

$$\begin{matrix} A & \cdot & X & = & B \end{matrix}$$

$$\begin{bmatrix} 5 & 7 \\ 3 & 8 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 11 \\ 18 \end{bmatrix}$$

coefficient matrix   variable matrix   constant matrix

The system of equations is now expressed as a **matrix equation**.

Example 1 Two-Variable Matrix Equation

Write a matrix equation for the system of equations.
$$5x - 6y = -47$$
$$3x + 2y = -17$$

Determine the coefficient, variable, and constant matrices.

$$5x - 6y = -47 \quad \rightarrow \quad \begin{bmatrix} 5 & -6 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \begin{bmatrix} -47 \\ -17 \end{bmatrix}$$
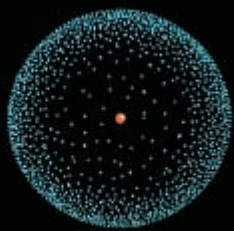$$3x + 2y = -17$$

Write the matrix equation.

$$\begin{matrix} A & \cdot & X & = & B \end{matrix}$$

$$\begin{bmatrix} 5 & -6 \\ 3 & 2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -47 \\ -17 \end{bmatrix}$$

You can use a matrix equation to determine the weight of an atom of an element.

Example 2 **Solve a Problem Using a Matrix Equation**

CHEMISTRY The molecular formula for glucose is $C_6H_{12}O_6$, which represents that a molecule of glucose has 6 carbon (C) atoms, 12 hydrogen (H) atoms, and 6 oxygen (O) atoms. One molecule of glucose weighs 180 atomic mass units (amu), and one oxygen atom weighs 16 atomic mass units. The formulas and weights for glucose and another sugar, sucrose, are listed below.

Sugar	Formula	Atomic Weight (amu)
glucose	$C_6H_{12}O_6$	180
sucrose	$C_{12}H_{22}O_{11}$	342

a. Write a system of equations that represents the weight of each atom.

Let c represent the weight of a carbon atom.
Let h represent the weight of a hydrogen atom.

Write an equation for the weight of each sugar. The subscript represents how many atoms of each element are in the molecule.

Glucose: $6c + 12h + 6(16) = 180$ Equation for glucose
             $6c + 12h + 96 = 180$ Simplify.
                  $6c + 12h = 84$ Subtract 96 from each side.

Sucrose: $12c + 22h + 11(16) = 342$ Equation for sucrose
             $12c + 22h + 176 = 342$ Simplify.
                  $12c + 22h = 166$ Subtract 176 from each side.

b. Write a matrix equation for the system of equations.

Determine the coefficient, variable, and constant matrices.

$$6c + 12h = 84 \atop 12c + 22h = 166 \rightarrow \begin{bmatrix} 6 & 12 \\ 12 & 22 \end{bmatrix} \begin{bmatrix} c \\ h \end{bmatrix} \begin{bmatrix} 84 \\ 166 \end{bmatrix}$$

Write the matrix equation.

$$\begin{matrix} A & \cdot & X & = & B \end{matrix}$$
$$\begin{bmatrix} 6 & 12 \\ 12 & 22 \end{bmatrix} \cdot \begin{bmatrix} c \\ h \end{bmatrix} = \begin{bmatrix} 84 \\ 166 \end{bmatrix}$$ You will solve this matrix equation in Exercise 11.

SOLVE SYSTEMS OF EQUATIONS You can solve a system of linear equations by solving a matrix equation. A matrix equation in the form $AX = B$, where A is a coefficient matrix, X is a variable matrix, and B is a constant matrix, can be solved in a similar manner as a linear equation of the form $ax = b$.

$ax = b$ Write the equation. $AX = B$

$\left(\frac{1}{a}\right)ax = \left(\frac{1}{a}\right)b$ Multiply each side by the inverse of the coefficient, if it exists. $A^{-1}AX = A^{-1}B$

$1x = \left(\frac{1}{a}\right)b$ $\left(\frac{1}{a}\right)a = 1, A^{-1}A = I$ $IX = A^{-1}B$

$x = \left(\frac{1}{a}\right)b$ $1x = x, IX = X$ $X = A^{-1}B$

Notice that the solution of the matrix equation is the product of the inverse of the coefficient matrix and the constant matrix.

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Example 3 Solve a System of Equations

Use a matrix equation to solve the system of equations.
$$6x + 2y = 11$$
$$3x - 8y = 1$$

The matrix equation is $\begin{bmatrix} 6 & 2 \\ 3 & -8 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 11 \\ 1 \end{bmatrix}$, when $A = \begin{bmatrix} 6 & 2 \\ 3 & -8 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$, and $B = \begin{bmatrix} 11 \\ 1 \end{bmatrix}$.

Step 1 Find the inverse of the coefficient matrix.
$$A^{-1} = \frac{1}{-48 - 6} \begin{bmatrix} -8 & -2 \\ -3 & 6 \end{bmatrix} \text{ or } -\frac{1}{54} \begin{bmatrix} -8 & -2 \\ -3 & 6 \end{bmatrix}$$

Step 2 Multiply each side of the matrix equation by the inverse matrix.

$$-\frac{1}{54} \begin{bmatrix} -8 & -2 \\ -3 & 6 \end{bmatrix} \cdot \begin{bmatrix} 6 & 2 \\ 3 & -8 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = -\frac{1}{54} \begin{bmatrix} -8 & -2 \\ -3 & 6 \end{bmatrix} \cdot \begin{bmatrix} 11 \\ 1 \end{bmatrix}$$ Multiply each side by A^{-1}

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = -\frac{1}{54} \begin{bmatrix} -90 \\ -27 \end{bmatrix}$$ Multiply matrices.

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{5}{3} \\ \frac{1}{2} \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

The solution is $\left(\frac{5}{3}, \frac{1}{2}\right)$. Check this solution in the original equation.

Example 4 System of Equations with No Solution

Use a matrix equation to solve the system of equations.
$$6a - 9b = -18$$
$$8a - 12b = 24$$

The matrix equation is $\begin{bmatrix} 6 & -9 \\ 8 & -12 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -18 \\ 24 \end{bmatrix}$, when $A = \begin{bmatrix} 6 & -9 \\ 8 & -12 \end{bmatrix}$, $X = \begin{bmatrix} a \\ b \end{bmatrix}$, and $B = \begin{bmatrix} -18 \\ 24 \end{bmatrix}$.
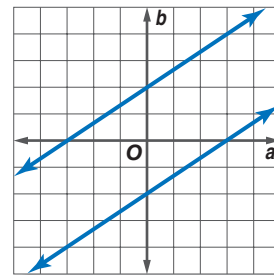
Find the inverse of the coefficient matrix.

$$A^{-1} = \frac{1}{-72 + 72} \begin{bmatrix} -12 & 9 \\ -8 & 6 \end{bmatrix}$$

The determinant of the coefficient matrix $\begin{bmatrix} 6 & -9 \\ 8 & -12 \end{bmatrix}$ is 0, so A^{-1} does not exist.

There is no unique solution of this system.

Graph the system of equations. Since the lines are parallel, this system has no solution. Therefore, the system is inconsistent.

To solve a system of equations with three variables, you can use the 3×3 identity matrix. However, finding the inverse of a 3×3 matrix may be tedious. Graphing calculators and computer programs offer fast and accurate methods for performing the necessary calculations.

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Graphing Calculator Investigation

Systems of Three Equations in Three Variables

You can use a graphing calculator and a matrix equation to solve systems of equations. Consider the system of equations below.

$3x - 2y + z = 0$

$2x + 3y - z = 17$

$5x - y + 4z = -7$

Think and Discuss

1. Write a matrix equation for the system of equations.

2. Enter the coefficient matrix as matrix A and the constant matrix as matrix B in the graphing calculator. Find the product of A^{-1} and B. Recall that the $\boxed{x^{-1}}$ key is used to find A^{-1}.

3. How is the result related to the solution?

Check for Understanding

Concept Check

1. **Write** the matrix equation $\begin{bmatrix} 2 & -3 \\ 1 & 4 \end{bmatrix} \cdot \begin{bmatrix} r \\ s \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$ as a system of linear equations.

2. **OPEN ENDED** Write a system of equations that does not have a unique solution.

3. **FIND THE ERROR** Tommy and Laura are solving a system of equations. They find that $A^{-1} = \begin{bmatrix} 3 & -2 \\ -7 & 5 \end{bmatrix}$, $B = \begin{bmatrix} -7 \\ -9 \end{bmatrix}$, and $X = \begin{bmatrix} x \\ y \end{bmatrix}$.

Tommy

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -7 & 5 \end{bmatrix} \cdot \begin{bmatrix} -7 \\ -9 \end{bmatrix}$

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$

Laura

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -7 \\ -9 \end{bmatrix} \cdot \begin{bmatrix} 3 & -2 \\ -7 & 5 \end{bmatrix}$

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 42 \\ 31 \end{bmatrix}$

Who is correct? Explain your reasoning.

Guided Practice

Write a matrix equation for each system of equations.

4. $x - y = -3$
   $x + 3y = 5$

5. $2g + 3h = 8$
   $-4g - 7h = -5$

6. $3a - 5b + 2c = 9$
   $4a + 7b + c = 3$
   $2a - c = 12$

Solve each matrix equation or system of equations by using inverse matrices.

7. $\begin{bmatrix} 3 & 1 \\ 4 & -2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 13 \\ 24 \end{bmatrix}$

8. $\begin{bmatrix} 8 & -1 \\ 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 16 \\ -9 \end{bmatrix}$

9. $5x - 3y = -30$
   $8x + 5y = 1$

10. $5s + 4t = 12$
    $4s - 3t = -1.25$

Application

11. **CHEMISTRY** Refer to Example 2 on page 203. Solve the system of equations to find the weight of a carbon, hydrogen, and oxygen atom.

CONTENTS

Write a matrix equation for each system of equations.

12. $3x - y = 0$
$x + 2y = -21$

13. $4x - 7y = 2$
$3x + 5y = 9$

14. $5a - 6b = -47$
$3a + 2b = -17$

15. $3m - 7n = -43$
$6m + 5n = -10$

16. $2a + 3b - 5c = 1$
$7a + 3c = 7$
$3a - 6b + c = -5$

17. $3x - 5y + 2z = 9$
$x - 7y + 3z = 11$
$4x - 3z = -1$

18. $x - y = 8$
$-2x - 5y - 6z = -27$
$9x + 10y - z = 54$

19. $3r - 5s + 6t = 21$
$11r - 12s + 16t = 15$
$-5r + 8s - 3t = -7$

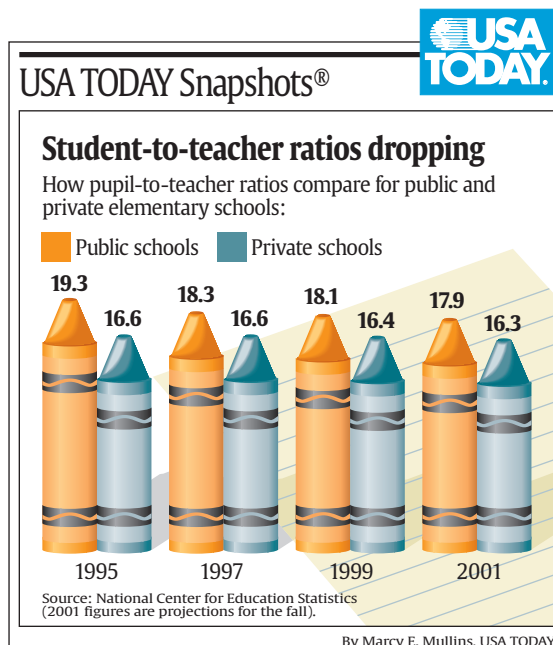**Solve each matrix equation or system of equations by using inverse matrices.**

20. $\begin{bmatrix} 7 & -3 \\ 2 & 5 \end{bmatrix} \cdot \begin{bmatrix} m \\ n \end{bmatrix} = \begin{bmatrix} 41 \\ 0 \end{bmatrix}$

21. $\begin{bmatrix} 3 & 1 \\ 2 & -1 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 13 \\ 2 \end{bmatrix}$

22. $\begin{bmatrix} 4 & -3 \\ 5 & 2 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -17 \\ -4 \end{bmatrix}$

23. $\begin{bmatrix} 7 & 1 \\ 3 & -8 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 43 \\ 10 \end{bmatrix}$

24. $\begin{bmatrix} 2 & -9 \\ 6 & 5 \end{bmatrix} \cdot \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 28 \\ -12 \end{bmatrix}$

25. $\begin{bmatrix} 6 & 5 \\ 3 & 2 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 18 \\ 7 \end{bmatrix}$

26. $6r + s = 9$
$3r = -2s$

27. $5a + 9b = -28$
$2a - b = -2$

28. $p - 2q = 1$
$p + 5q = 22$

29. $4m - 7n = -63$
$3m + 2n = 18$

30. $x + 2y = 8$
$3x + 2y = 6$

31. $4x - 3y = 5$
$2x + 9y = 6$

32. PILOT TRAINING Hai-Ling is training for his pilot's license. Flight instruction costs $105 per hour, and the simulator costs $45 per hour. The school requires students to spend 4 more hours in airplane training than in the simulator. If Hai-Ling can afford to spend $3870 on training, how many hours can he spend training in an airplane and in a simulator?

33. SCHOOLS The graphic shows that student-to-teacher ratios are dropping in both public and private schools. If these rates of change remain constant, predict when the student-to-teacher ratios for private and public schools will be the same.

34. CHEMISTRY Cara is preparing an acid solution. She needs 200 milliliters of 48% concentration solution. Cara has 60% and 40% concentration solutions in her lab. How many milliliters of 40% acid solution should be mixed with 60% acid solution to make the required amount of 48% acid solution?

USA TODAY Snapshots®

Student-to-teacher ratios dropping

How pupil-to-teacher ratios compare for public and private elementary schools:

Public schools ■ Private schools

	1995	1997	1999	2001
Public	19.3	18.3	18.1	17.9
Private	16.6	16.6	16.4	16.3

Source: National Center for Education Statistics (2001 figures are projections for the fall).

By Marcy E. Mullins, USA TODAY

CONTENTS

35. CRITICAL THINKING Describe the solution set of a system of equations if the coefficient matrix does not have an inverse.

36. WRITING IN MATH   Answer the question that was posed at the beginning of the lesson.

How can matrices be used in population ecology?

Include the following in your answer:
- a system of equations that can be used to find the number of each species the region can support, and
- a solution of the problem using matrices.

37. Solve the system of equations $6a + 8b = 5$ and $10a - 12b = 2$.

Ⓐ $\left(\frac{3}{4}, \frac{1}{2}\right)$ 　　　 Ⓑ $\left(\frac{1}{2}, \frac{3}{4}\right)$ 　　　 Ⓒ $\left(\frac{1}{2}, -\frac{1}{2}\right)$ 　　　 Ⓓ $\left(\frac{1}{2}, \frac{1}{4}\right)$

38. SHORT RESPONSE The Yogurt Shoppe sells cones in three sizes: small $0.89; medium, $1.19; and large, $1.39. One day Scott sold 52 cones. He sold seven more medium cones than small cones. If he sold $58.98 in cones, how many of each size did he sell?

Graphing Calculator

INVERSE MATRICES Use a graphing calculator to solve each system of equations using inverse matrices.

39. $2a - b + 4c = 6$
$a + 5b - 2c = -6$
$3a - 2b + 6c = 8$

40. $3x - 5y + 2z = 22$
$2x + 3y - z = -9$
$4x + 3y + 3z = 1$

41. $2q + r + s = 2$
$-q - r + 2s = 7$
$-3q + 2r + 3s = 7$

Maintain Your Skills

Mixed Review Find the inverse of each matrix, if it exists.   *(Lesson 4-7)*

42. $\begin{bmatrix} 4 & 4 \\ 2 & 3 \end{bmatrix}$ 　　　 **43.** $\begin{bmatrix} 9 & 5 \\ 7 & 4 \end{bmatrix}$ 　　　 **44.** $\begin{bmatrix} -3 & -6 \\ 5 & 10 \end{bmatrix}$

Use Cramer's Rule to solve each system of equations.   *(Lesson 4-6)*

45. $6x + 7y = 10$
$3x - 4y = 20$

46. $6a + 7b = -10.15$
$9.2a - 6b = 69.944$

47. $\frac{x}{2} - \frac{2y}{3} = 2\frac{1}{3}$
$3x + 4y = -50$

48. ECOLOGY If you recycle a $3\frac{1}{2}$-foot stack of newspapers, one less 20-foot loblolly pine tree will be needed for paper. Use a prediction equation to determine how many feet of loblolly pine trees will *not* be needed for paper if you recycle a pile of newspapers 20 feet tall.   *(Lesson 2-5)*

Solve each equation. Check your solutions.   *(Lesson 1-4)*

49. $|x - 3| = 7$ 　　　 **50.** $-4|d + 2| = -12$ 　　　 **51.** $5|k - 4| = k + 8$

WebQuest **Internet Project**

Lessons in Home Buying, Selling

It is time to complete your project. Use the information and data you have gathered about home buying and selling to prepare a portfolio or Web page. Be sure to include your tables, graphs, and calculations. You may also wish to include additional data, information, or pictures.

www.algebra2.com/webquest

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Augmented Matrices

Using a TI-83 Plus, you can solve a system of linear equations using the **MATRX** function. An **augmented matrix** contains the coefficient matrix with an extra column containing the constant terms. The reduced row echelon function of a graphing calculator reduces the augmented matrix so that the solution of the system of equations can be easily determined.

Write an augmented matrix for the following system of equations. Then solve the system by using the reduced row echelon form on the graphing calculator.

$3x + y + 3z = 2$
$2x + y + 2z = 1$
$4x + 2y + 5z = 5$

Step 1 Write the augmented matrix and enter it into a calculator.

The augmented matrix $B = \begin{bmatrix} 3 & 1 & 3 & 2 \\ 2 & 1 & 2 & 1 \\ 4 & 2 & 5 & 5 \end{bmatrix}$.

Begin by entering the matrix.

KEYSTROKES: *Review matrices on page 163.*

Step 2 Find the reduced row echelon form (rref) using the graphing calculator.

KEYSTROKES: 2nd [MATRX] ▶ ALPHA [B]

2nd [MATRX] 2) ENTER

Study the reduced echelon matrix. The first three columns are the same as a 3×3 identity matrix. The first row represents $x = -2$, the second row represents $y = -1$, and the third row represents $z = 3$. The solution is $(-2, -1, 3)$.

```
rref([B])
      [[1 0 0 -2]
       [0 1 0 -1]
       [0 0 1 3 ]]
```
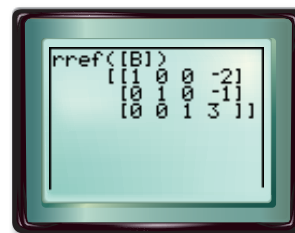
Exercises

Write an augmented matrix for each system of equations. Then solve with a graphing calculator.

1. $x - 3y = 5$
$2x + y = 1$

2. $15x + 11y = 36$
$4x - 3y = -26$

3. $2x + y = 5$
$2x - 3y = 1$

4. $3x - y = 0$
$2x - 3y = 1$

5. $3x - 2y + z = -2$
$x - y + 3z = 5$
$-x + y + z = -1$

6. $x - y + z = 2$
$x - z = 1$
$y + 2z = 0$

www.algebra2.com/other_calculator_keystrokes

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Study Guide and Review

Vocabulary and Concept Check

column matrix (p. 155)	expansion by minors (p. 183)	minor (p. 183)	second-order determinant (p. 182)
Cramer's Rule (p. 189)	identity matrix (p. 195)	preimage (p. 175)	square matrix (p. 155)
determinant (p. 182)	image (p. 175)	reflection (p. 177)	third-order determinant (p. 183)
dilation (p. 176)	inverse (p. 195)	rotation (p. 178)	transformation (p. 175)
dimension (p. 155)	isometry (p. 175)	row matrix (p. 155)	translation (p. 175)
element (p. 155)	matrix (p. 154)	scalar (p. 162)	vertex matrix (p. 175)
equal matrices (p. 155)	matrix equation (p. 202)	scalar multiplication (p. 162)	zero matrix (p. 155)

Choose the correct term to complete each sentence.

1. The matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is a(n) _____ for multiplication.

2. When an image and a preimage are congruent, then the transformation is called a(n) _____.

3. _____ is the process of multiplying a matrix by a constant.

4. A(n) _____ is when a figure is moved around a center point.

5. The _____ of $\begin{bmatrix} -1 & 4 \\ 2 & -3 \end{bmatrix}$ is -5.

6. A(n) _____ is the product of the coefficient matrix and the variable matrix equal to the constant matrix.

7. The _____ of a matrix tell how many rows and columns are in the matrix.

8. A(n) _____ occurs when a figure is moved from one location to another on the coordinate plane.

9. The matrices $\begin{bmatrix} 3x \\ x + 2y \end{bmatrix}$ and $\begin{bmatrix} y \\ 7 \end{bmatrix}$ are _____ if $x = 1$ and $y = 3$.

10. A(n) _____ is when a geometric figure is enlarged or reduced.

determinant

dilation

dimensions

equal matrices

identity matrix

isometry

matrix equation

rotation

scalar multiplication

translation

Lesson-by-Lesson Review

4-1 Introduction to Matrices

See pages 154–158.

Concept Summary

- A matrix is a rectangular array of variables or constants in horizontal rows and vertical columns.
- Equal matrices have the same dimensions and corresponding elements equal.

Example Solve $\begin{bmatrix} 2x \\ y \end{bmatrix} = \begin{bmatrix} 32 + 6y \\ 7 - x \end{bmatrix}$ for x and y.

Since the matrices are equal, corresponding elements are equal. You can write two linear equations.

$2x = 32 + 6y$
$y = 7 - x$

(continued on the next page)

CONTENTS

Solve the system of equations.

$2x = 32 + 6y$	First equation
$2x = 32 + 6(7 - x)$	Substitute $7 - x$ for y.
$2x = 32 + 42 - 6x$	Distributive Property
$8x = 74$	Add $6x$ to each side.
$x = 9.25$	Divide each side by 8.

The solution is $(9.25, -2.25)$.

To find the value for y, substitute 9.25 for x in either equation.

$y = 7 - x$	Second equation
$= 7 - 9.25$	Substitute 9.25 for x.
$= -2.25$	Simplify.

Exercises Solve each equation. *See Example 3 on pages 155 and 156.*

11. $\begin{bmatrix} 2y - x \\ x \end{bmatrix} = \begin{bmatrix} 3 \\ 4y - 1 \end{bmatrix}$

12. $\begin{bmatrix} 7x \\ x + y \end{bmatrix} = \begin{bmatrix} 5 + 2y \\ 11 \end{bmatrix}$

13. $\begin{bmatrix} 3x + y \\ x - 3y \end{bmatrix} = \begin{bmatrix} -3 \\ -1 \end{bmatrix}$

14. $\begin{bmatrix} 2x - y \\ 6x - y \end{bmatrix} = \begin{bmatrix} 2 \\ 22 \end{bmatrix}$

4-2 Operations with Matrices

See pages 160–166.

Concept Summary

- Matrices can be added or subtracted if they have the same dimensions. Add or subtract corresponding elements.
- To multiply a matrix by a scalar k, multiply each element in the matrix by k.

Examples **1** Find $A - B$ if $A = \begin{bmatrix} 3 & 8 \\ -5 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} -4 & 6 \\ 1 & 9 \end{bmatrix}$.

$A - B = \begin{bmatrix} 3 & 8 \\ -5 & 2 \end{bmatrix} - \begin{bmatrix} -4 & 6 \\ 1 & 9 \end{bmatrix}$ Definition of matrix subtraction

$= \begin{bmatrix} 3 - (-4) & 8 - 6 \\ -5 - 1 & 2 - 9 \end{bmatrix}$ Subtract corresponding elements.

$= \begin{bmatrix} 7 & 2 \\ -6 & -7 \end{bmatrix}$ Simplify.

2 If $X = \begin{bmatrix} 3 & 2 & -1 \\ 4 & -6 & 0 \end{bmatrix}$, find $4X$.

$4X = 4\begin{bmatrix} 3 & 2 & -1 \\ 4 & -6 & 0 \end{bmatrix}$

$= \begin{bmatrix} 4(3) & 4(2) & 4(-1) \\ 4(4) & 4(-6) & 4(0) \end{bmatrix}$ or $\begin{bmatrix} 12 & 8 & -4 \\ 16 & -24 & 0 \end{bmatrix}$ Multiply each element by 4.

Exercises Perform the indicated matrix operations. If the matrix does not exist, write *impossible*. *See Examples 1, 2, and 4 on pages 160–162.*

15. $\begin{bmatrix} -4 & 3 \\ -5 & 2 \end{bmatrix} + \begin{bmatrix} 1 & -3 \\ 3 & -8 \end{bmatrix}$

16. $[0.2 \quad 1.3 \quad -0.4] - [2 \quad 1.7 \quad 2.6]$

17. $\begin{bmatrix} 1 & -5 \\ -2 & 3 \end{bmatrix} + \dfrac{3}{4}\begin{bmatrix} 0 & 4 \\ -16 & 8 \end{bmatrix}$

18. $\begin{bmatrix} 1 & 0 & -3 \\ 4 & -5 & 2 \end{bmatrix} - 2\begin{bmatrix} -2 & 3 & 5 \\ -3 & -1 & 2 \end{bmatrix}$

CONTENTS

4-3 Multiplying Matrices

Concept Summary

- Two matrices can be multiplied if and only if the number of columns in the first matrix is equal to the number of rows in the second matrix.

Example Find XY if $X = [6 \quad 4 \quad 1]$ and $Y = \begin{bmatrix} 2 & 5 \\ -3 & 0 \\ -1 & 3 \end{bmatrix}$.

$XY = [6 \quad 4 \quad 1] \cdot \begin{bmatrix} 2 & 5 \\ -3 & 0 \\ -1 & 3 \end{bmatrix}$ 　　　　Write an equation.

$\quad = [6(2) + 4(-3) + 1(-1) \quad 6(5) + 4(0) + 1(3)]$ 　Multiply columns by rows.

$\quad = [-1 \quad 33]$ 　　　　Simplify.

Exercises Find each product, if possible. *See Example 2 on page 168.*

19. $[2 \quad 7] \cdot \begin{bmatrix} 5 \\ -4 \end{bmatrix}$ 　　　　**20.** $\begin{bmatrix} 8 & -3 \\ 6 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & -3 \\ 1 & -5 \end{bmatrix}$

21. $\begin{bmatrix} 3 & 4 \\ 1 & 0 \\ 2 & -5 \end{bmatrix} \cdot \begin{bmatrix} -2 & 4 & 5 \\ 3 & 0 & -1 \\ 1 & 0 & -1 \end{bmatrix}$ 　　**22.** $\begin{bmatrix} 3 & 0 & -1 \\ 4 & -2 & 3 \end{bmatrix} \cdot \begin{bmatrix} 7 & 1 \\ 6 & -3 \\ 2 & 1 \end{bmatrix}$

4-4 Transformations with Matrices

Concept Summary

- Use matrix addition and a translation matrix to find the coordinates of a translated figure.
- Use scalar multiplication to perform dilations.
- To reflect a figure, multiply the vertex matrix on the left by a reflection matrix.

reflection over　　　　　reflection over　　　　　reflection over

x-axis: $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ 　　y-axis: $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ 　　line $y = x$: $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

- To rotate a figure counterclockwise about the origin, multiply the vertex matrix on the left by a rotation matrix.

90° rotation: $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ 　180° rotation: $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ 　270° rotation: $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$
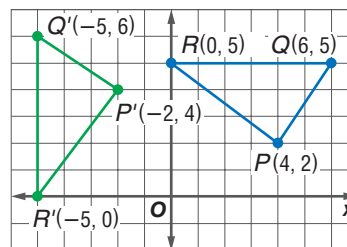
Example **Find the coordinates of the vertices of the image of △PQR with P(4, 2), Q(6, 5), and R(0, 5) after it is rotated 90° counterclockwise about the origin.**

Write the ordered pairs in a vertex matrix. Then multiply the vertex matrix by the rotation matrix.

$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 4 & 6 & 0 \\ 2 & 5 & 5 \end{bmatrix} = \begin{bmatrix} -2 & -5 & -5 \\ 4 & 6 & 0 \end{bmatrix}$

The coordinates of the vertices of △P′Q′R′ are P′(−2, 4), Q′(−5, 6), and R′(−5, 0).

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Exercises   For Exercises 23–26, use the figure at the right.
See Examples 1–5 on pages 175–178.

23. Find the coordinates of the image after a translation
4 units right and 5 units down.

24. Find the coordinates of the image of the figure after
a dilation by a scale factor of 2.

25. Find the coordinates of the image after a reflection over
the *y*-axis.

26. Find the coordinates of the image of the figure after a
rotation of 180°.

A(−3, 5)   *B(4, 3)*   *O*   *x*   *C(−1, −2)*
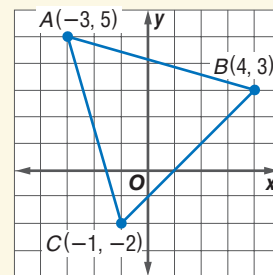
4-5   *Determinants*

Concept Summary

- Determinant of a 2 × 2 matrix: $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

- Determinant of a 3 × 3 matrix: $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$

- Area of a triangle with vertices at (a, b), (c, d), and (e, f):

$$|A| \text{ where } A = \frac{1}{2}\begin{vmatrix} a & b & 1 \\ c & d & 1 \\ e & f & 1 \end{vmatrix}$$

Examples   **1**   Find the value of $\begin{vmatrix} 3 & 6 \\ -4 & 2 \end{vmatrix}$.

$\begin{vmatrix} 3 & 6 \\ -4 & 2 \end{vmatrix} = 3(2) - (-4)(6)$ Definition of determinant

$\quad\quad\quad = 6 - (-24) \text{ or } 30$ Simplify.

2   Evaluate $\begin{vmatrix} 3 & 1 & 5 \\ 1 & -2 & 1 \\ 0 & -1 & 2 \end{vmatrix}$ using expansion by minors.

$\begin{vmatrix} 3 & 1 & 5 \\ 1 & -2 & 1 \\ 0 & -1 & 2 \end{vmatrix} = 3\begin{vmatrix} -2 & 1 \\ -1 & 2 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} + 5\begin{vmatrix} 1 & -2 \\ 0 & -1 \end{vmatrix}$ Expansion by minors

$\quad\quad\quad = 3(-4 - (-1)) - 1(2 - 0) + 5(-1 - 0)$ Evaluate 2 × 2 determinants.

$\quad\quad\quad = -9 - 2 - 5 \text{ or } -16$ Simplify.

Exercises   Find the value of each determinant.   *See Examples 1–3 on pages 182–184.*

27. $\begin{vmatrix} 4 & 11 \\ -7 & 8 \end{vmatrix}$ **28.** $\begin{vmatrix} 6 & -7 \\ 5 & 3 \end{vmatrix}$ **29.** $\begin{vmatrix} 12 & 8 \\ 9 & 6 \end{vmatrix}$

30. $\begin{vmatrix} 2 & -3 & 1 \\ 0 & 7 & 8 \\ 2 & 1 & 3 \end{vmatrix}$ **31.** $\begin{vmatrix} 7 & -4 & 5 \\ 1 & 3 & -6 \\ 5 & -1 & -2 \end{vmatrix}$ **32.** $\begin{vmatrix} 6 & 3 & -2 \\ -4 & 2 & 5 \\ -3 & -1 & 0 \end{vmatrix}$

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4-6 Cramer's Rule

Concept Summary

- Cramer's Rule for two variables:
  The solution of the system of equations $ax + by = e$ and $cx + dy = f$
  is (x, y), where $x = \dfrac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$, $y = \dfrac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$, and $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$.

- Cramer's Rule for three variables:
  The solution of the system whose equations are $ax + by + cz = j$,
  $dx + ey + fz = k$, $gx + hy + iz = \ell$ is (x, y, z), where

  $$x = \frac{\begin{vmatrix} j & b & c \\ k & e & f \\ \ell & h & i \end{vmatrix}}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} a & j & c \\ d & k & f \\ g & \ell & i \end{vmatrix}}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}}, \quad z = \frac{\begin{vmatrix} a & b & j \\ d & e & k \\ g & h & \ell \end{vmatrix}}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}}, \quad \text{and} \quad \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \neq 0.$$

Example Use Cramer's Rule to solve each system of equations $5a - 3b = 7$ and $3a + 9b = -3$.

$a = \dfrac{\begin{vmatrix} 7 & -3 \\ -3 & 9 \end{vmatrix}}{\begin{vmatrix} 5 & -3 \\ 3 & 9 \end{vmatrix}}$ Cramer's Rule $b = \dfrac{\begin{vmatrix} 5 & 7 \\ 3 & -3 \end{vmatrix}}{\begin{vmatrix} 5 & -3 \\ 3 & 9 \end{vmatrix}}$

$= \dfrac{63 - 9}{45 + 9}$ Evaluate each determinant. $= \dfrac{-15 - 21}{45 + 9}$

$= \dfrac{54}{54}$ or 1 Simplify. $= \dfrac{-36}{54}$ or $-\dfrac{2}{3}$

The solution is $\left(1, -\dfrac{2}{3}\right)$.

Exercises Use Cramer's Rule to solve each system of equations.
See Examples 1 and 3 on pages 190 and 191.

33. $9a - b = 1$
$\ 3a + 2b = 12$

34. $x + 5y = 14$
$\ -2x + 6y = 4$

35. $3x + 4y = -15$
$\ 2x - 7y = 19$

36. $8a + 5b = 2$
$\ -6a - 4b = -1$

37. $6x - 7z = 13$
$\ 8y + 2z = 14$
$\ 7x + z = 6$

38. $2a - b - 3c = -20$
$\ 4a + 2b + c = 6$
$\ 2a + b - c = -6$

4-7 Identity and Inverse Matrices

Concept Summary

- An identity matrix is a square matrix with ones on the diagonal and zeros in the other positions.

- Two matrices are inverses of each other if their product is the identity matrix.

- The inverse of matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $A^{-1} \dfrac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, where $ad - bc \neq 0$.

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Chapter

4 For More ...

• Extra Practice, see pages 834–836.
• Mixed Problem Solving, see page 865.

Example Find the inverse of $S = \begin{bmatrix} 3 & -4 \\ 2 & 1 \end{bmatrix}$.

Find the value of the determinant.

$$\begin{vmatrix} 3 & -4 \\ 2 & 1 \end{vmatrix} = 3 - (-8) \text{ or } 11$$

Use the formula for the inverse matrix.

$$S^{-1} = \frac{1}{11} \begin{bmatrix} 1 & 4 \\ -2 & 3 \end{bmatrix}$$

Exercises Find the inverse of each matrix, if it exists. *See Example 2 on page 197.*

39. $\begin{bmatrix} 3 & 2 \\ 4 & -2 \end{bmatrix}$

40. $\begin{bmatrix} 8 & 6 \\ 9 & 7 \end{bmatrix}$

41. $\begin{bmatrix} 2 & 4 \\ -3 & 6 \end{bmatrix}$

42. $\begin{bmatrix} 6 & -2 \\ 3 & -1 \end{bmatrix}$

43. $\begin{bmatrix} 0 & 2 \\ 5 & -4 \end{bmatrix}$

44. $\begin{bmatrix} 6 & -1 & 0 \\ 5 & 8 & -2 \end{bmatrix}$

4-8 *Using Matrices to Solve Systems of Equations*

See pages
202–207.

Concept Summary

• A system of equations can be written as a matrix equation in the form
$A \cdot X = B$.

$$\begin{array}{l} 2x + 3y = 12 \\ x - 4y = 6 \end{array} \rightarrow \begin{bmatrix} 2 & 3 \\ 1 & -4 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 12 \\ 6 \end{bmatrix}$$

• To solve a matrix equation, find the inverse of the coefficient matrix. Then multiply each side by the inverse matrix, so $X = A^{-1}B$.

Example Solve $\begin{bmatrix} 4 & 8 \\ 2 & -3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 12 \\ 13 \end{bmatrix}$.

Step 1 Find the inverse of the coefficient matrix.

$$A^{-1} = \frac{1}{-12 - 16} \begin{bmatrix} -3 & -8 \\ -2 & 4 \end{bmatrix} \text{ or } -\frac{1}{28} \begin{bmatrix} -3 & -8 \\ -2 & 4 \end{bmatrix}$$

Step 2 Multiply each side by the inverse matrix.

$$-\frac{1}{28} \begin{bmatrix} -3 & -8 \\ -2 & 4 \end{bmatrix} \cdot \begin{bmatrix} 4 & 8 \\ 2 & -3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = -\frac{1}{28} \begin{bmatrix} -3 & -8 \\ -2 & 4 \end{bmatrix} \cdot \begin{bmatrix} 12 \\ 13 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = -\frac{1}{28} \begin{bmatrix} -140 \\ 28 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \end{bmatrix}$$

The solution is $(5, -1)$.

Exercises Solve each matrix equation or system of equations by using inverse matrices. *See Example 3 on page 204.*

45. $\begin{bmatrix} 5 & -2 \\ 1 & 3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 16 \\ 10 \end{bmatrix}$

46. $\begin{bmatrix} 4 & 1 \\ 3 & -2 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 9 \\ 4 \end{bmatrix}$

47. $\begin{array}{l} 3x + 8 = -y \\ 4x - 2y = -14 \end{array}$

48. $\begin{array}{l} 3x - 5y = -13 \\ 4x + 3y = 2 \end{array}$

CONTENTS

Vocabulary and Concepts

Choose the letter that best matches each description.

1. $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

2. $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix}$

3. $\dfrac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

a. inverse of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

b. determinant of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

c. matrix equation for $ax + by = e$ and $cx + dy = f$

Skills and Applications

Solve each equation.

4. $\begin{bmatrix} 3x + 1 \\ 2y \end{bmatrix} = \begin{bmatrix} 10 \\ 4 + y \end{bmatrix}$

5. $\begin{bmatrix} 2x & y + 1 \\ 13 & -2 \end{bmatrix} = \begin{bmatrix} -16 & -7 \\ 13 & z - 8 \end{bmatrix}$

Perform the indicated matrix operations. If the matrix does not exist, write *impossible*.

6. $\begin{bmatrix} 2 & -4 & 1 \\ 3 & 8 & -2 \end{bmatrix} - 2\begin{bmatrix} 1 & 2 & -4 \\ -2 & 3 & 7 \end{bmatrix}$

7. $\begin{bmatrix} 1 & 6 & 7 \\ 1 & -3 & -4 \end{bmatrix} \cdot \begin{bmatrix} -4 & 3 \\ -1 & -2 \\ 2 & 5 \end{bmatrix}$

Find the value of each determinant.

8. $\begin{vmatrix} -1 & 4 \\ -6 & 3 \end{vmatrix}$

9. $\begin{vmatrix} 5 & -3 & 2 \\ -6 & 1 & 3 \\ -1 & 4 & -7 \end{vmatrix}$

Find the inverse of each matrix, if it exists.

10. $\begin{bmatrix} -2 & 5 \\ 3 & 1 \end{bmatrix}$

11. $\begin{bmatrix} -6 & -3 \\ 8 & 4 \end{bmatrix}$

12. $\begin{bmatrix} 5 & -2 \\ 6 & 3 \end{bmatrix}$

Solve each matrix equation or system of equations by using inverse matrices.

13. $\begin{bmatrix} 1 & 8 \\ 2 & -6 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3 \\ -17 \end{bmatrix}$

14. $\begin{bmatrix} 5 & 7 \\ -9 & 3 \end{bmatrix} \cdot \begin{bmatrix} m \\ n \end{bmatrix} = \begin{bmatrix} 41 \\ -105 \end{bmatrix}$

15. $5a + 2b = -49$
    $2a + 9b = 5$

For Exercises 16–18, use $\triangle ABC$ whose vertices have coordinates $A(6, 3)$, $B(1, 5)$, and $C(-1, 4)$.

16. Use the determinant to find the area of $\triangle ABC$.

17. Translate $\triangle ABC$ so that the coordinates of B' are $(3, 1)$. What are the coordinates of A' and C'?

18. Find the coordinates of the vertices of a similar triangle whose perimeter is five times that of $\triangle ABC$.

19. **RETAIL SALES** Brittany is preparing boxes of assorted chocolates. Chocolate-covered peanuts cost $7 per pound. Chocolate-covered caramels cost $6.50 per pound. The boxes of assorted candies contain five more pounds of peanut candies than caramel candies. If the total amount sold was $575, how many pounds of each candy were needed to make the boxes?

20. **STANDARDIZED TEST PRACTICE** If $\begin{bmatrix} 43 & z \\ 7x - 2 & 2x + 3 \end{bmatrix} = \begin{bmatrix} z + 3 & 2m + 5 \\ y & 37 \end{bmatrix}$, then $y =$
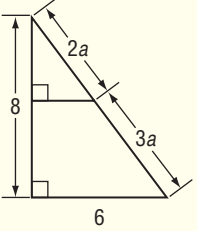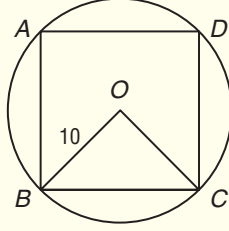
Ⓐ 120.

Ⓑ 117.

Ⓒ 22.

Ⓓ not enough information

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CONTENTS

Part 1 Multiple Choice

**Record your answers on the answer sheet provided by your teacher or on a sheet of paper.**

1. If the average (arithmetic mean) of ten numbers is 18 and the average of six of these numbers is 12, what is the average of the other four numbers?

  Ⓐ 15   Ⓑ 18

  Ⓒ 27   Ⓓ 28

2. A car travels 65 miles per hour for 2 hours. A truck travels 60 miles per hour for 1.5 hours. What is the difference between the number of miles traveled by the car and the number of miles traveled by the truck?

  Ⓐ 31.25   Ⓑ 40

  Ⓒ 70   Ⓓ 220

3. In the figure, $a =$

  Ⓐ 1.

  Ⓑ 2.

  Ⓒ 3.

  Ⓓ 4.

4. If the circumference of a circle is $\frac{4\pi}{3}$, then what is half of its area?

  Ⓐ $\frac{2\pi}{9}$ Ⓑ $\frac{4\pi}{9}$

  Ⓒ $\frac{8\pi}{9}$ Ⓓ $\frac{2\pi^2}{9}$

5. A line is represented by the equation $x = 6$. What is the slope of the line?

  Ⓐ 0   Ⓑ $\frac{5}{6}$

  Ⓒ 6   Ⓓ undefined

6. In the figure, $ABCD$ is a square inscribed in the circle centered at O. If $\overline{OB}$ is 10 units long, how many units long is minor arc BC?

  Ⓐ $\frac{5}{2}\pi$ units

  Ⓑ 5π units

  Ⓒ 10π units

  Ⓓ 20π units

7. If $3 < x < 5 < y < 10$, then which of the following best defines $\frac{x}{y}$?

  Ⓐ $\frac{3}{10} < \frac{x}{y} < 1$

  Ⓑ $\frac{3}{10} < \frac{x}{y} < \frac{1}{2}$

  Ⓒ $\frac{3}{5} < \frac{x}{y} < \frac{1}{2}$

  Ⓓ $\frac{3}{5} < \frac{x}{y} < 1$

8. If $x + 3y = 12$ and $\frac{2}{3}x - y = 5$, then $x =$

  Ⓐ 1.   Ⓑ 8.

  Ⓒ 9.   Ⓓ 13.5.

9. At what point do the two lines with the equations $7x - 3y = 13$ and $y = 2x - 3$ intersect?

  Ⓐ $(-4, -11)$ Ⓑ $(4, 11)$

  Ⓒ $(4, 5)$ Ⓓ $(5, 4)$

10. If $N = \begin{bmatrix} -1 & 0 \\ 5 & -2 \end{bmatrix}$ and $M = \begin{bmatrix} -1 & 0 \\ 5 & 2 \end{bmatrix}$, find $N - M$.

  Ⓐ $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ Ⓑ $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

  Ⓒ $\begin{bmatrix} 0 & 0 \\ 0 & -4 \end{bmatrix}$ Ⓓ $\begin{bmatrix} -2 & 0 \\ 0 & -4 \end{bmatrix}$

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Part 2 Short Response/Grid In

Record your answers on the answer sheet provided by your teacher or on a sheet of paper.

11. A computer manufacturer reduced the price of its Model X computer by 3%. If the new price of the Model X computer is $2489, then how much did the computer cost, in dollars, before its price was reduced? (Round to the nearest dollar.)

12. In square $PQRS$, $PQ = 4$, $PU = UQ$, and $PT = TS$. What is the area of the shaded region?
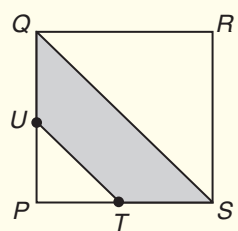
13. A rectangular solid has two faces the same size and shape as Figure 1 and four faces the same size and shape as Figure 2. What is the volume of the solid in cubic units?

Figure 1   Figure 2
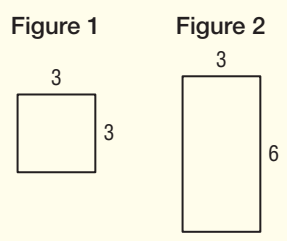
14. If the average (arithmetic mean) of three different positive integers is 60, what is the greatest possible value of one of the integers?

15. The perimeter of a triangle is 15. The lengths of the sides are integers. If the length of one side is 6, what is the shortest possible length of another side of the triangle?

The Princeton Review Test-Taking Tip

Questions 14, 15 Watch for the phrases "greatest possible" or "least possible." Think logically about the conditions that make an expression greatest or least. Notice what types of numbers are used—positive, even, prime, integers.

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16. In this sequence below, each term after the first term is $\frac{1}{4}$ of the term preceding it. What is the sixth term of this sequence?
$$320, 80, 20, \dots$$

17. If the sum of two numbers is 5 and their difference is 2, what is their product?

Part 3 Quantitative Comparison

Compare the quantity in Column A and the quantity in Column B. Then determine whether:

(A) **the quantity in Column A is greater,**

(B) **the quantity in Column B is greater,**

(C) **the two quantities are equal, or**

(D) **the relationship cannot be determined from the information given.**

Column A	Column B
18. $xy = 0$	
y	0
19. 4, 8, 16, 18	
the greatest of the numbers listed above which is the sum of two equal even integers	the greatest of the numbers listed above which is the sum of two equal odd integers
20.	
the volume of a cube with edges 4 inches long	the sum of the volumes of eight cubes each having edges 2 units long

21. Point P with coordinates (x, y) is exactly 4 units from the origin.

x	y

22.
$$r + s + t = 30$$
$$r + s - t = 8$$

t	11

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