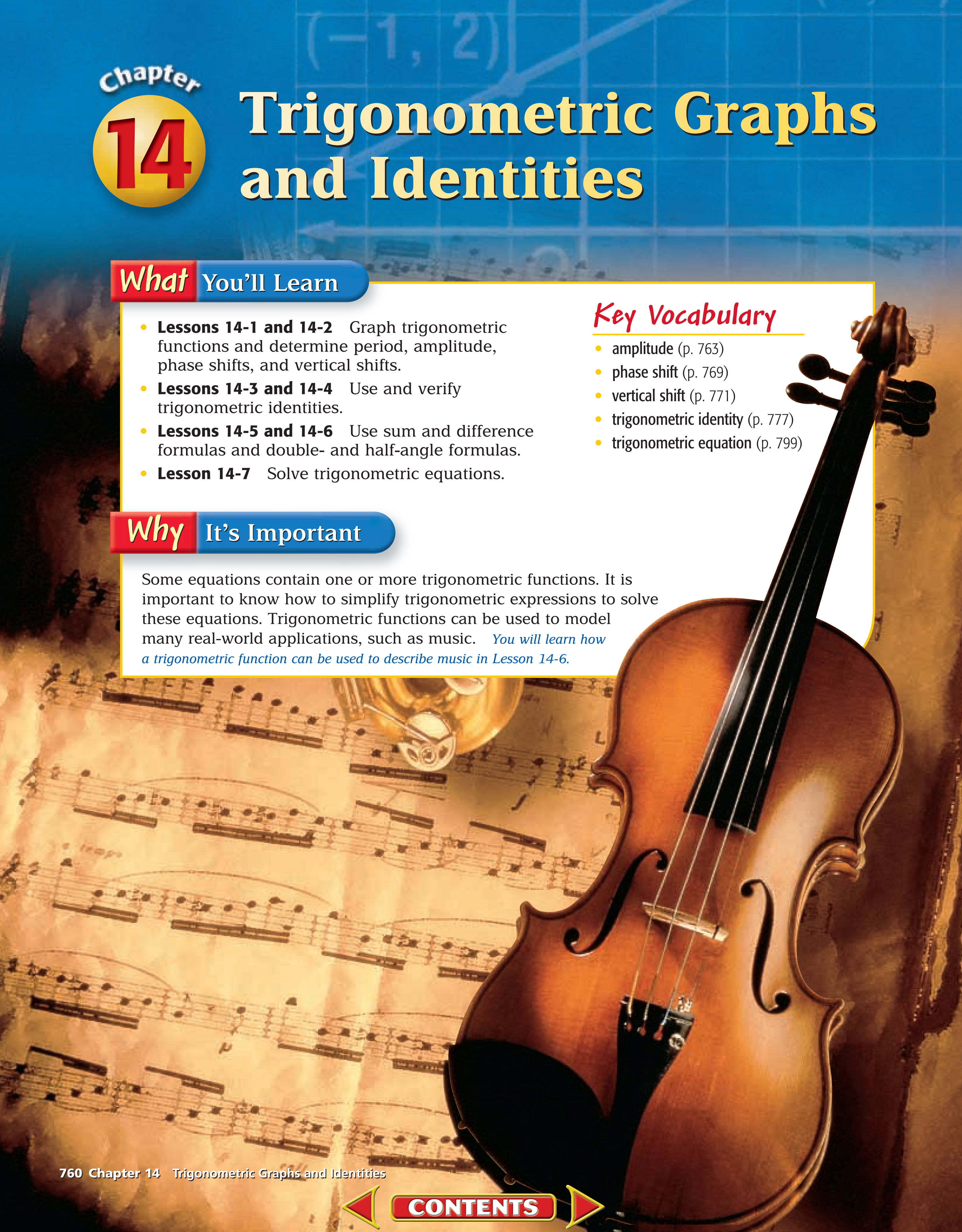


Chapter 14

Trigonometric Graphs and Identities

What You'll Learn

- **Lessons 14-1 and 14-2** Graph trigonometric functions and determine period, amplitude, phase shifts, and vertical shifts.
- **Lessons 14-3 and 14-4** Use and verify trigonometric identities.
- **Lessons 14-5 and 14-6** Use sum and difference formulas and double- and half-angle formulas.
- **Lesson 14-7** Solve trigonometric equations.

Key Vocabulary

- amplitude (p. 763)
- phase shift (p. 769)
- vertical shift (p. 771)
- trigonometric identity (p. 777)
- trigonometric equation (p. 799)

Why It's Important

Some equations contain one or more trigonometric functions. It is important to know how to simplify trigonometric expressions to solve these equations. Trigonometric functions can be used to model many real-world applications, such as music. *You will learn how a trigonometric function can be used to describe music in Lesson 14-6.*

CONTENTS

Getting Started

Prerequisite Skills To be successful in this chapter, you'll need to master these skills and be able to apply them in problem-solving situations. Review these skills before beginning Chapter 14.

For Lessons 14-1 and 14-2 — Trigonometric Values

Find the exact value of each trigonometric function. *(For review, see Lesson 13-3.)*

1. $\sin 135°$ **2.** $\tan 315°$ **3.** $\cos 90°$ **4.** $\tan 45°$

5. $\sin \frac{5\pi}{4}$ **6.** $\cos \frac{7\pi}{6}$ **7.** $\sin \frac{11\pi}{6}$ **8.** $\tan \frac{3\pi}{2}$

For Lessons 14-3, 14-5, and 14-6 — Circular Functions

Find the exact value of each trigonometric function. *(For review, see Lesson 13-6.)*

9. $\cos(-150°)$ **10.** $\sin 510°$ **11.** $\cot \frac{9\pi}{4}$ **12.** $\sec \frac{13\pi}{6}$

13. $\tan\left(-\frac{3\pi}{2}\right)$ **14.** $\csc(-720°)$ **15.** $\cos \frac{7\pi}{3}$ **16.** $\tan \frac{8\pi}{3}$

For Lesson 14-4 — Factor Polynomials

Factor completely. If the polynomial is not factorable, write *prime*. *(For review, see Lesson 5-4.)*

17. $-15x^2 - 5x$ **18.** $2x^4 - 4x^2$ **19.** $x^3 + 4$

20. $x^2 - 6x + 8$ **21.** $2x^2 - 3x - 2$ **22.** $3x^3 - 2x^2 - x$

For Lesson 14-7 — Solve Quadratic Equations

Solve each equation by factoring. *(For review, see Lesson 6-3.)*

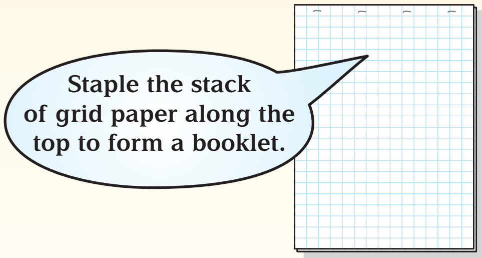
23. $x^2 - 5x - 24 = 0$ **24.** $x^2 - 2x - 48 = 0$ **25.** $x^2 + 3x - 40 = 0$

26. $x^2 - 12x = 0$ **27.** $-2x^2 - 11x - 12 = 0$ **28.** $x^2 - 16 = 0$

FOLDABLES™ Study Organizer

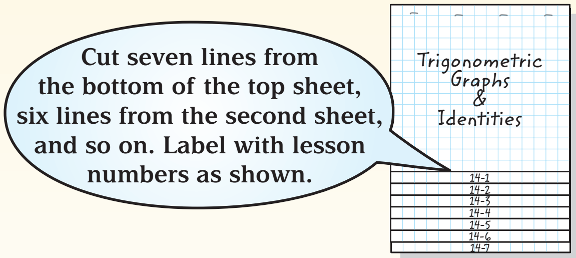
Make this Foldable to help you organize information about trigonometric graphs and identities. Begin with eight sheets of grid paper.

Step 1 Staple



Staple the stack of grid paper along the top to form a booklet.

Step 2 Cut and Label



Cut seven lines from the bottom of the top sheet, six lines from the second sheet, and so on. Label with lesson numbers as shown.

Reading and Writing As you read and study the chapter, use each page to write notes and to graph examples for each lesson.

14-1 Graphing Trigonometric Functions

What You'll Learn

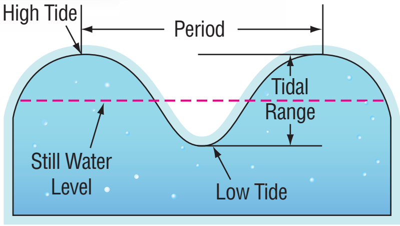
- Graph trigonometric functions.
- Find the amplitude and period of variation of the sine, cosine, and tangent functions.

Vocabulary

- amplitude

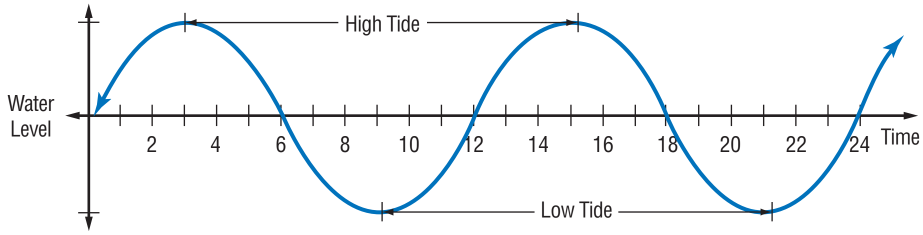
Why can you predict the behavior of tides?

The rise and fall of tides can have great impact on the communities and ecosystems that depend upon them. One type of tide is a semidiurnal tide. This means that bodies of water, like the Atlantic Ocean, have two high tides and two low tides a day. Because tides are periodic, they behave the same way each day.



GRAPH TRIGONOMETRIC FUNCTIONS

The diagram below illustrates the water level as a function of time for a body of water with semidiurnal tides.



Study Tip

Look Back

To review **period** and **periodic functions**, see Lesson 13-6.

In each cycle of high and low tides, the pattern repeats itself. Recall that a function whose graph repeats a basic pattern is called a *periodic function*.

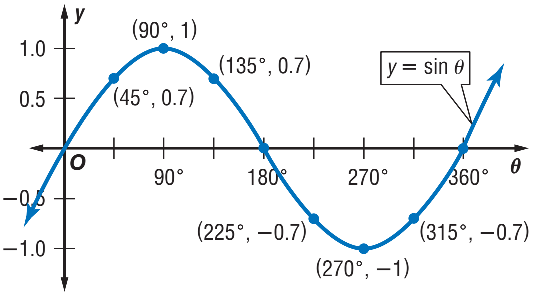
To find the period, start from any point on the graph and proceed to the right until the pattern begins to repeat. The simplest approach is to begin at the origin. Notice that after about 12 hours the graph begins to repeat. Thus, the period of the function is about 12 hours.

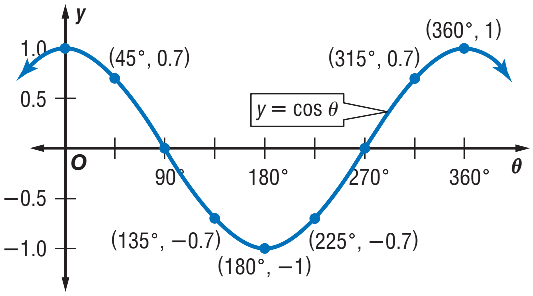
To graph the periodic functions $y = \sin\theta$, $y = \cos\theta$, or $y = \tan\theta$, use values of θ expressed either in degrees or radians. Ordered pairs for points on these graphs are of the form $(\theta, \sin\theta)$, $(\theta, \cos\theta)$, and $(\theta, \tan\theta)$, respectively.

θ	0°	30°	45°	60°	90°	120°	135°	150°	180°	210°	225°	240°	270°	300°	315°	330°	360°
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0
nearest tenth	0	0.5	0.7	0.9	1	0.9	0.7	0.5	0	−0.5	−0.7	−0.9	−1	−0.9	−0.7	−0.5	0
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
nearest tenth	1	0.9	0.7	0.5	0	−0.5	−0.7	−0.9	−1	−0.9	−0.7	−0.5	0	0.5	0.7	0.9	1
$\tan\theta$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	nd	$-\sqrt{3}$	-1	$-\frac{\sqrt{3}}{3}$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	nd	$-\sqrt{3}$	-1	$-\frac{\sqrt{3}}{3}$	0
nearest tenth	0	0.6	1	1.7	nd	−1.7	−1	−0.6	0	0.6	1	1.7	nd	−1.7	−1	−0.6	0
θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π

nd = not defined

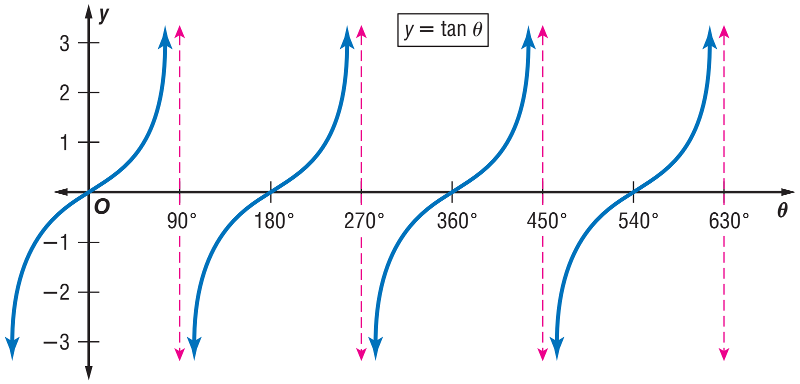
After plotting several points, complete the graphs of $y = \sin \theta$ and $y = \cos \theta$ by connecting the points with a smooth, continuous curve. Recall from Chapter 13 that each of these functions has a period of 360° or 2π radians. That is, the graph of each function repeats itself every 360° or 2π radians.





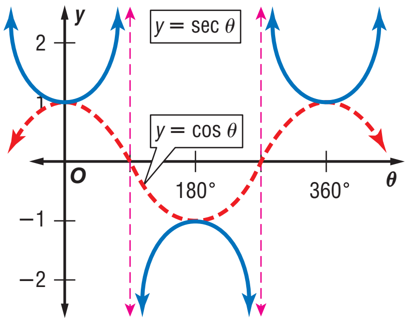
Notice that both the sine and cosine have a maximum value of 1 and a minimum value of −1. The **amplitude** of the graph of a periodic function is the absolute value of half the difference between its maximum value and its minimum value. So, for both the sine and cosine functions, the amplitude of their graphs is $\left|\frac{1-(-1)}{2}\right|$ or 1.

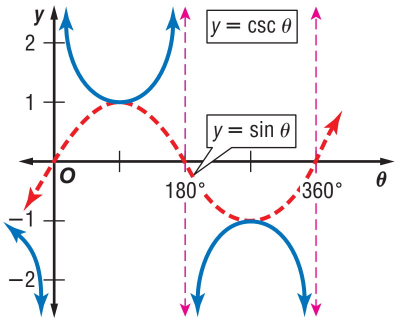
The graph of the tangent function can also be drawn by plotting points. By examining the values for $\tan \theta$ in the table, you can see that the tangent function is not defined for 90°, 270°, …, 90° + $k \cdot 180°$, where k is an integer. The graph is separated by vertical asymptotes whose x-intercepts are the values for which $y = \tan \theta$ is not defined.

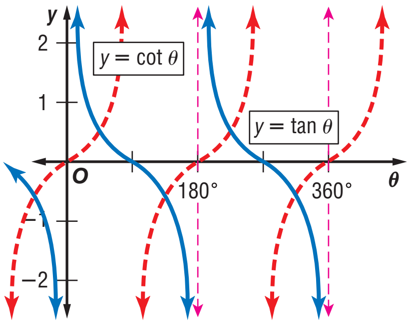


The period of the tangent function is 180° or π radians. Since the tangent function has no maximum or minimum value, it has no amplitude.

The graphs of the secant, cosecant, and cotangent functions are shown below. Compare them to the graphs of the cosine, sine, and tangent functions, which are shown in red.







Notice that the period of the secant and cosecant functions is 360° or 2π radians. The period of the cotangent is 180° or π radians. Since none of these functions have a maximum or minimum value, they have no amplitude.

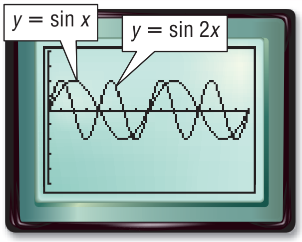
VARIATIONS OF TRIGONOMETRIC FUNCTIONS Just as with other functions, a trigonometric function can be used to form a family of graphs by changing the period and amplitude.

Graphing Calculator Investigation

Period and Amplitude

On a TI-83 Plus graphing calculator, set the MODE to degrees.

Think and Discuss



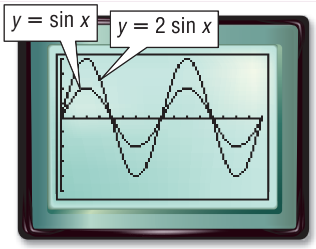
[0, 720] scl: 45 by [−2.5, 2.5] scl: 0.5

1. Graph $y = \sin x$ and $y = \sin 2x$. What is the maximum value of each function?

2. How many times does each function reach a maximum value?

3. Graph $y = \sin\left(\frac{x}{2}\right)$. What is the maximum value of this function? How many times does this function reach its maximum value?

4. Use the equations $y = \sin bx$ and $y = \cos bx$. Repeat Exercises 1–3 for maximum values and the other values of b. What conjecture can you make about the effect of b on the maximum values and the periods of these functions?



[0, 720] scl: 45 by [−2.5, 2.5] scl: 0.5

5. Graph $y = \sin x$ and $y = 2 \sin x$. What is the maximum value of each function? What is the period of each function?

6. Graph $y = \frac{1}{2} \sin x$. What is the maximum value of this function? What is the period of this function?

7. Use the equations $y = a \sin x$ and $y = a \cos x$. Repeat Exercises 5 and 6 for other values of a. What conjecture can you make about the effect of a on the amplitudes and periods of $y = a \sin x$ and $y = a \cos x$?

Study Tip

Amplitude and Period

Note that the amplitude affects the graph along the vertical axis and the period affects it along the horizontal axis.

The results of the investigation suggest the following generalization.

Key Concept — Amplitudes and Periods

- **Words** For functions of the form $y = a \sin b\theta$ and $y = a \cos b\theta$, the amplitude is $|a|$, and the period is $\frac{360°}{|b|}$ or $\frac{2\pi}{|b|}$.

  For functions of the form $y = a \tan b\theta$, the amplitude is not defined, and the period is $\frac{180°}{|b|}$ or $\frac{\pi}{|b|}$.

- **Examples**

  $y = 3 \sin 4\theta$ amplitude 3 and period $\frac{360°}{4}$ or 90°

  $y = -6 \cos 5\theta$ amplitude $|-6|$ or 6 and period $\frac{2\pi}{5}$

  $y = 2 \tan \frac{1}{3}\theta$ no amplitude and period 3π

You can use the amplitude and period of a trigonometric function to help you graph the function.

Example 1 Graph Trigonometric Functions

Find the amplitude and period of each function. Then graph the function.

a. $y = \cos 3\theta$

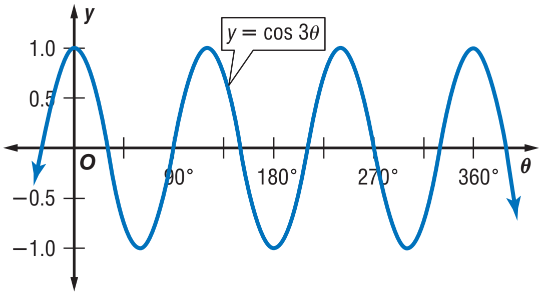
First, find the amplitude.

$|a| = |1|$ The coefficient of cos 3θ is 1.

Next, find the period.

$$\frac{360°}{|b|} = \frac{360°}{|3|} \quad b = 3$$
$$= 120°$$

Use the amplitude and period to graph the function.



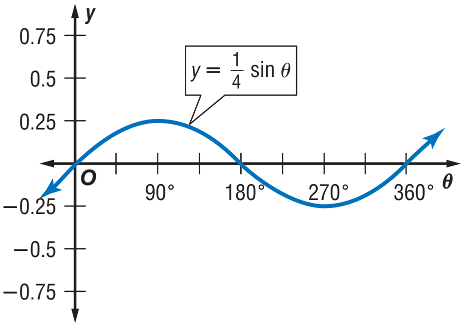
b. $y = \frac{1}{4} \sin \theta$

Amplitude: $|a| = \left|\frac{1}{4}\right|$

$= \frac{1}{4}$

Period: $\frac{360°}{|b|} = \frac{360°}{|1|}$

$= 360°$



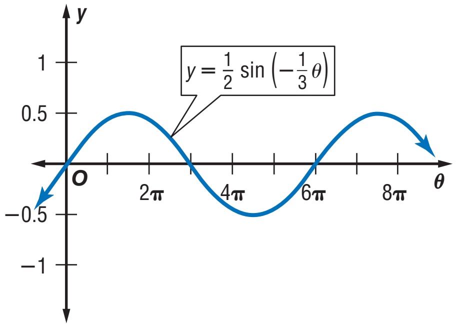
c. $y = \frac{1}{2} \sin \left(-\frac{1}{3}\theta\right)$

Amplitude: $|a| = \left|\frac{1}{2}\right|$

$= \frac{1}{2}$

Period: $\frac{2\pi}{|b|} = \frac{2\pi}{\left|-\frac{1}{3}\right|}$

$= 6\pi$



You can use trigonometric functions to describe real-world situations.

Example 2 Use Trigonometric Functions

OCEANOGRAPHY **Refer to the application at the beginning of the lesson. Suppose the tidal range of a city on the Atlantic coast is 18 feet. A tide is at *equilibrium* when it is at its normal level, halfway between its highest and lowest points.**

a. Write a function to represent the height h of the tide. Assume that the tide is at equilibrium at $t = 0$ and that the high tide is beginning.

Since the height of the tide is 0 at $t = 0$, use the sine function $h = a \sin bt$, where a is the amplitude of the tide and t is the time in hours.

Find the amplitude. The difference between high tide and low tide is the tidal range or 18 feet.

$$a = \frac{18}{2} \text{ or } 9$$

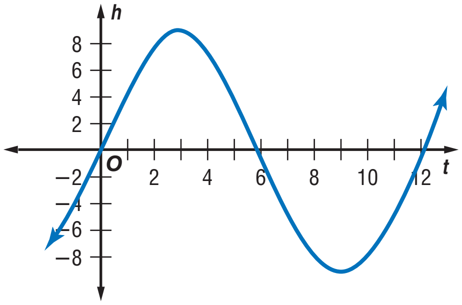
Find the value of b. Each tide cycle lasts about 12 hours.

$$\frac{2\pi}{|b|} = 12 \qquad \text{period} = \frac{2\pi}{|b|}$$

$$b = \frac{2\pi}{12} \text{ or } \frac{\pi}{6} \qquad \text{Solve for } b.$$

Thus, an equation to represent the height of the tide is $h = 9 \sin \frac{\pi}{6}t$.

b. Graph the tide function.



More About. . .

Oceanography

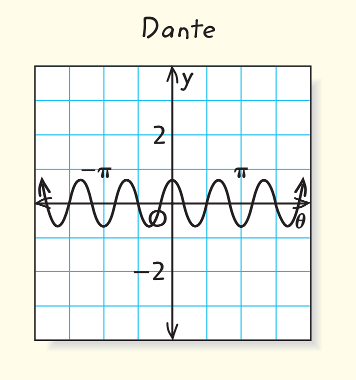
Lake Superior has one of the smallest tidal ranges. It can be measured in inches, while the tidal range in the Bay of Funday in Canada measures up to 50 feet.

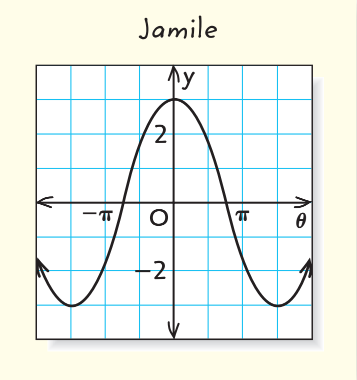
Source: Office of Naval Research

Check for Understanding

Concept Check

1. **OPEN ENDED** Explain why $y = \tan \theta$ has no amplitude.
2. **Explain** what it means to say that the period of a function is 180°.
3. **FIND THE ERROR** Dante and Jamile graphed $y = 3 \cos \frac{2}{3}\theta$.





Who is correct? Explain your reasoning.

CONTENTS

Guided Practice **Find the amplitude, if it exists, and period of each function. Then graph each function.**

4. $y = \frac{1}{2}\sin\theta$

5. $y = 2\sin\theta$

6. $y = \frac{2}{3}\cos\theta$

7. $y = \frac{1}{4}\tan\theta$

8. $y = \csc 2\theta$

9. $y = 4\sin 2\theta$

10. $y = 4\cos\frac{3}{4}\theta$

11. $y = \frac{1}{2}\sec 3\theta$

12. $y = \frac{3}{4}\cos\frac{1}{2}\theta$

Application **BIOLOGY** **For Exercises 13 and 14, use the following information.**
In a certain wildlife refuge, the population of field mice can be modeled by $y = 3000 + 1250\sin\frac{\pi}{6}t$, where y represents the number of mice and t represents the number of months past March 1 of a given year.

13. Determine the period of the function. What does this period represent?

14. What is the maximum number of mice and when does this occur?

Practice and Apply

Homework Help

For Exercises	See Examples
15–35	1
36–41	2

Extra Practice

See page 859.

Find the amplitude, if it exists, and period of each function. Then graph each function.

15. $y = 3\sin\theta$

16. $y = 5\cos\theta$

17. $y = 2\csc\theta$

18. $y = 2\tan\theta$

19. $y = \frac{1}{5}\sin\theta$

20. $y = \frac{1}{3}\sec\theta$

21. $y = \sin 4\theta$

22. $y = \sin 2\theta$

23. $y = \sec 3\theta$

24. $y = \cot 5\theta$

25. $y = 4\tan\frac{1}{3}\theta$

26. $y = 2\cot\frac{1}{2}\theta$

27. $y = 6\sin\frac{2}{3}\theta$

28. $y = 3\cos\frac{1}{2}\theta$

29. $y = 3\csc\frac{1}{2}\theta$

30. $y = \frac{1}{2}\cot 2\theta$

31. $2y = \tan\theta$

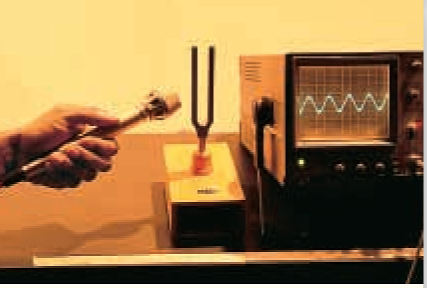
32. $\frac{3}{4}y = \frac{2}{3}\sin\frac{3}{5}\theta$

33. Draw a graph of a sine function with an amplitude $\frac{3}{5}$ and a period of 90°. Then write an equation for the function.

34. Draw a graph of a cosine function with an amplitude of $\frac{7}{8}$ and a period of $\frac{2\pi}{5}$. Then write an equation for the function.

35. **COMMUNICATIONS** The carrier wave for a certain FM radio station can be modeled by the equation $y = A\sin(10^7 \cdot 2\pi t)$, where A is the amplitude of the wave and t is the time in seconds. Determine the period of the carrier wave.

More About. . .



Medicine

The tuning fork was invented in 1711 by English trumpeter John Shore.

Source: www.encarta.msn.com

MEDICINE **For Exercises 36 and 37, use the following information.**
Doctors may use a tuning fork that resonates at a given frequency as an aid to diagnose hearing problems. The sound wave produced by a tuning fork can be modeled using a sine function.

36. If the amplitude of the sine function is 0.25, write the equations for tuning forks that resonate with a frequency of 64, 256, and 512 Hertz.

37. How do the periods of the tuning forks compare?

38. **CRITICAL THINKING** A function is called *even* if the graphs of $y = f(x)$ and $y = f(-x)$ are exactly the same. Which of the six trigonometric functions are even? Justify your answer with a graph of each function.

BOATING **For Exercises 39–41, use the following information.**
A marker buoy off the coast of Gulfport, Mississippi, bobs up and down with the waves. The distance between the highest and lowest point is 4 feet. The buoy moves from its highest point to its lowest point and back to its highest point every 10 seconds.

39. Write an equation for the motion of the buoy. Assume that it is at equilibrium at $t = 0$ and that it is on the way up from the normal water level.

40. Draw a graph showing the height of the buoy as a function of time.

41. What is the height of the buoy after 12 seconds?

42. **WRITING IN MATH** Answer the question that was posed at the beginning of the lesson.

Why can you predict the behavior of tides?

Include the following in your answer:

- an explanation of why certain tidal characteristics follow the patterns seen in the graph of the sine function, and
- a description of how to determine the amplitude of a function using the maximum and minimum values.

Standardized Test Practice

43. What is the period of $f(x) = \frac{1}{2} \cos 3x$?

Ⓐ 120° Ⓑ 180° Ⓒ 360° Ⓓ 720°

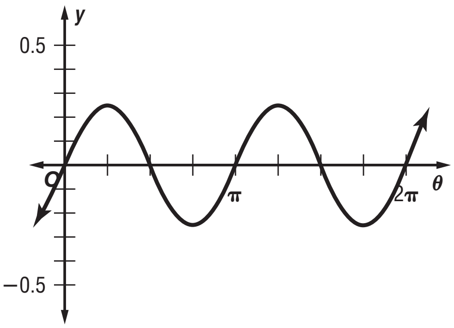
44. Identify the equation of the graphed function.

Ⓐ $y = \frac{1}{2} \sin 4\theta$

Ⓑ $y = 2 \sin \frac{1}{4}\theta$

Ⓒ $y = \frac{1}{4} \sin 2\theta$

Ⓓ $y = 4 \sin \frac{1}{2}\theta$



Maintain Your Skills

Mixed Review

Solve each equation. *(Lesson 13-7)*

45. $x = \text{Sin}^{-1} 1$ **46.** $\text{Arcsin}\ (-1) = y$ **47.** $\text{Arccos}\ \frac{\sqrt{2}}{2} = x$

Find the exact value of each function. *(Lesson 13-6)*

48. sin 390° **49.** sin (−315°) **50.** cos 405°

51. **PROBABILITY** There are 8 girls and 8 boys on the Faculty Advisory Board. Three are juniors. Find the probability of selecting a boy or a girl from the committee who is not a junior. *(Lesson 12-5)*

52. Find the first five terms of the sequence in which $a_1 = 3, a_{n+1} = 2a_n + 5$. *(Lesson 11-5)*

Getting Ready for the Next Lesson

PREREQUISITE SKILL **Graph each pair of functions on the same set of axes.**
*(To review **graphs of quadratic functions**, see Lesson 6-6.)*

53. $y = x^2, y = 3x^2$ **54.** $y = 3x^2, y = 3x^2 - 4$

55. $y = 2x^2, y = 2(x + 1)^2$ **56.** $y = x^2 + 2, y = (x - 3)^2 + 2$

CONTENTS

14-2 Translations of Trigonometric Graphs

What You'll Learn

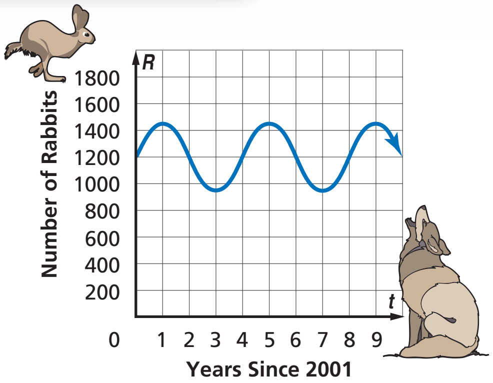
- Graph horizontal translations of trigonometric graphs and find phase shifts.
- Graph vertical translations of trigonometric graphs.

Vocabulary

- phase shift
- vertical shift
- midline

How can translations of trigonometric graphs be used to show animal populations?

In predator-prey ecosystems, the number of predators and the number of prey tend to vary in a periodic manner. In a certain region with coyotes as predators and rabbits as prey, the rabbit population R can be modeled by the equation $R = 1200 + 250 \sin \frac{1}{2}\pi t$, where t is the time in years since January 1, 2001.



HORIZONTAL TRANSLATIONS Recall that a translation is a type of transformation in which the image is identical to the preimage in all aspects except its location on the coordinate plane. A horizontal translation shifts to the left or right, and not upward or downward.

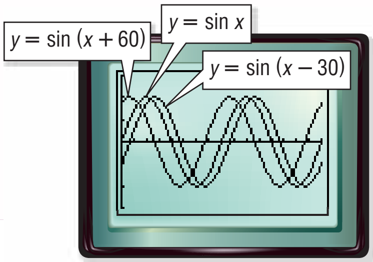
Graphing Calculator Investigation

Horizontal Translations

On a TI-83 Plus, set the MODE to degrees.

Think and Discuss

1. Graph $y = \sin x$ and $y = \sin (x - 30)$. How do the two graphs compare?
2. Graph $y = \sin (x + 60)$. How does this graph compare to the other two?
3. What conjecture can you make about the effect of h in the function $y = \sin (x - h)$?
4. Test your conjecture on the following pairs of graphs.
    - $y = \cos x$ and $y = \cos (x + 30)$
    - $y = \tan x$ and $y = \tan (x - 45)$
    - $y = \sec x$ and $y = \sec (x + 75)$



[0, 720] scl: 45 by [−1.5, 1.5] scl: 0.5

Notice that when a constant is added to an angle measure in a trigonometric function, the graph is shifted to the left or to the right. If (x, y) are coordinates of $y = \sin x$, then $(x \pm h, y)$ are coordinates of $y = \sin (x \mp h)$. A horizontal translation of a trigonometric function is called a **phase shift**.

CONTENTS

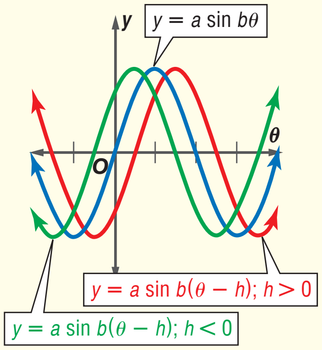
Key Concept

Phase Shift

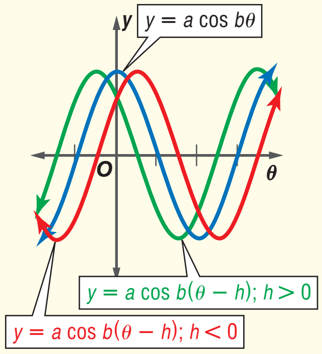
- **Words:** The phase shift of the functions $y = a \sin b(\theta - h)$, $y = a \cos b(\theta - h)$, and $y = a \tan b(\theta - h)$ is h, where $b > 0$.

  If $h > 0$, the shift is to the right. If $h < 0$, the shift is to the left.
- **Models:**

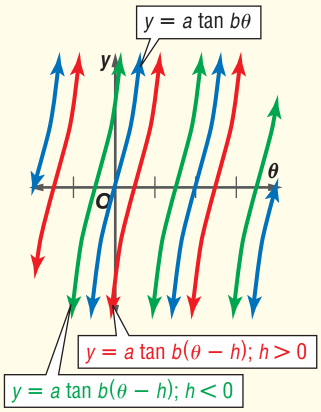
Sine



Cosine



Tangent



The secant, cosecant, and cotangent can be graphed using the same rules.

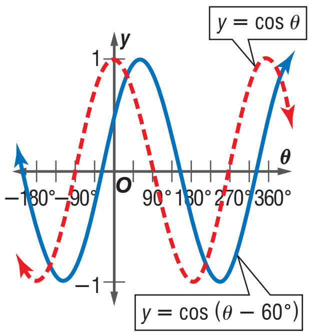
Example 1 Graph Horizontal Translations

State the amplitude, period, and phase shift for each function. Then graph the function.

a. $y = \cos(\theta - 60°)$

Since $a = 1$ and $b = 1$, the amplitude and period of the function are the same as $y = \cos\theta$. However, $h = 60°$, so the phase shift is 60°. Because $h > 0$, the parent graph is shifted to the right.

To graph $y = \cos(\theta - 60°)$, consider the graph of $y = \cos\theta$. Graph this function and then shift the graph 60° to the right. The graph $y = \cos(\theta - 60°)$ is the graph of $y = \cos\theta$ shifted to the right.



Study Tip

Verifying a Graph

After drawing the graph of a trigonometric function, select value of θ and evaluate them in the equation to verify your graph.

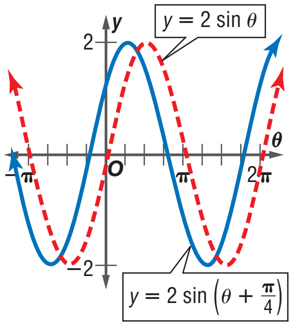
b. $y = 2\sin\left(\theta + \frac{\pi}{4}\right)$

Amplitude: $a = |2|$ or 2

Period: $\frac{2\pi}{|b|} = \frac{2\pi}{|1|}$ or 2π

Phase Shift: $h = -\frac{\pi}{4}$ $\left(\theta + \frac{\pi}{4}\right) = \theta - \left(-\frac{\pi}{4}\right)$

The phase shift is to the left since $-\frac{\pi}{4} < 0$.



CONTENTS

Study Tip

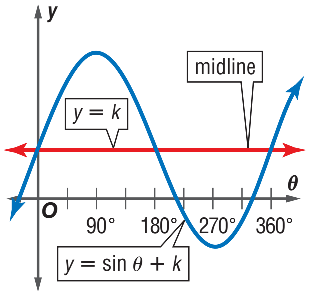
Look Back

Pay close attention to trigonometric functions for the placement of parentheses. Note that $\sin(\theta + x) \neq \sin\theta + x$. The expression on the left represents a phase shift while the expression on the right represents a vertical shift.

VERTICAL TRANSLATIONS In Chapter 6, you learned that the graph of $y = x^2 + 4$ is a vertical translation of the parent graph of $y = x^2$. Similarly, graphs of trigonometric functions can be translated vertically through a **vertical shift**.

When a constant is added to a trigonometric function, the graph is shifted upward or downward. If (x, y) are coordinates of $y = \sin x$, then $(x, y \pm k)$ are coordinates of $y = \sin x \pm k$.

A new horizontal axis called the **midline** becomes the reference line about which the graph oscillates. For the graph of $y = \sin\theta + k$, the midline is the graph of $y = k$.



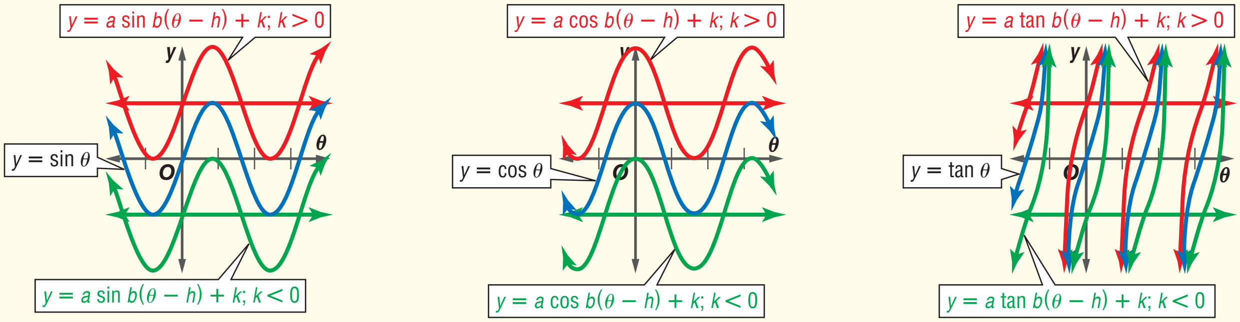
Key Concept

Vertical Shift

- **Words** The vertical shift of the functions $y = a \sin b(\theta - h) + k$, $y = a \cos b(\theta - h) + k$, and $y = a \tan b(\theta - h) + k$ is k.

  If $k > 0$, the shift is up. If $k < 0$, the shift is down. The midline is $y = k$.

- **Models:**



The secant, cosecant, and cotangent can be graphed using the same rules.

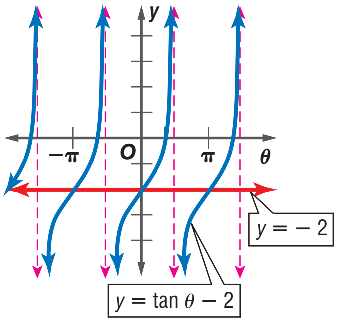
Example 2 Graph Vertical Translations

State the vertical shift, equation of the midline, amplitude, and period for each function. Then graph the function.

a. $y = \tan\theta - 2$

Since $\tan\theta - 2 = \tan\theta + (-2)$, $k = -2$, and the vertical shift is -2. Draw the midline, $y = -2$. The tangent function has no amplitude and the period is the same as that of $\tan\theta$.

Draw the graph of the function relative to the midline.



Study Tip

Look Back

It may be helpful to first graph the parent graph $y = \sin\theta$ in one color. Then apply the vertical shift and graph the function in another color. Then apply the change in amplitude and graph the function in the final color.

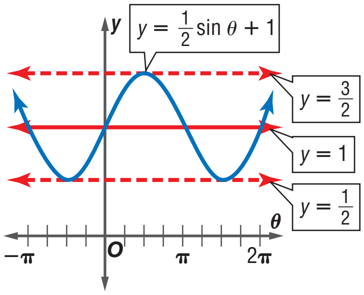
b. $y = \frac{1}{2}\sin\theta + 1$

Vertical shift: $k = 1$, so the midline is the graph of $y = 1$.

Amplitude: $|a| = \left|\frac{1}{2}\right|$ or $\frac{1}{2}$

Period: $\frac{2\pi}{|b|} = 2\pi$

Since the amplitude of the function is $\frac{1}{2}$, draw dashed lines parallel to the midline that are $\frac{1}{2}$ unit above and below the midline. Then draw the sine curve.



In general, use the following steps to graph any trigonometric function.

Concept Summary — *Graphing Trigonometric Functions*

Step 1 Determine the vertical shift, and graph the midline.

Step 2 Determine the amplitude, if it exists. Use dashed lines to indicate the maximum and minimum values of the function.

Step 3 Determine the period of the function and graph the appropriate function.

Step 4 Determine the phase shift and translate the graph accordingly.

Example 3 Graph Transformations

State the vertical shift, amplitude, period, and phase shift of $y = 4\cos\left[\frac{1}{2}\left(\theta - \frac{\pi}{3}\right)\right] - 6$. Then graph the function.

The function is written in the form $y = a\cos[b(\theta - h)] + k$. Identify the values of k, a, b, and h.

$k = -6$, so the vertical shift is -6.

$a = 4$, so the amplitude is $|4|$ or 4.

$b = \frac{1}{2}$, so the period is $\frac{2\pi}{\left|\frac{1}{2}\right|}$ or 4π.

$h = \frac{\pi}{3}$, so the phase shift is $\frac{\pi}{3}$ to the right.

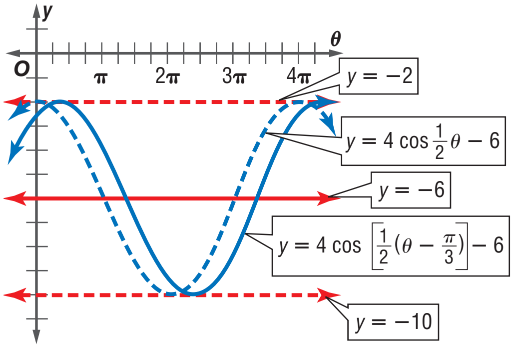
Then graph the function.

Step 1 The vertical shift is -6. Graph the midline $y = -6$.

Step 2 The amplitude is 4. Draw dashed lines 4 units above and below the midline at $y = -2$ and $y = -10$.

Step 3 The period is 4π, so the graph will be stretched. Graph $y = 4\cos\frac{1}{2}\theta - 6$ using the midline as a reference.

Step 4 Shift the graph $\frac{\pi}{3}$ to the right.



You can use information about amplitude, period, and translations of trigonometric functions to model real-world applications.

Example 4 Use Translations to Solve a Problem

HEALTH **Suppose a person's resting blood pressure is 120 over 80. This means that the blood pressure oscillates between a maximum of 120 and a minimum of 80. If this person's resting heart rate is 60 beats per minute, write a sine function that represents the blood pressure for t seconds. Then graph the function.**

Explore You know that the function is periodic and can be modeled using sine.

Plan Let P represent blood pressure and let t represent time in seconds. Use the equation $P = a \sin [b(t - h)] + k$.

Solve

- Write the equation for the midline. Since the maximum is 120 and the minimum is 80, the midline lies halfway between these values.

$$P = \frac{120 + 80}{2} \text{ or } 100$$

- Determine the amplitude by finding the difference between the midline value and the maximum and minimum values.

$$a = |120 - 100| \qquad a = |80 - 100|$$
$$= |20| \text{ or } 20 \qquad = |-20| \text{ or } 20$$

Thus, $a = 20$.

- Determine the period of the function and solve for b. Recall that the period of a function can be found using the expression $\frac{2\pi}{|b|}$. Since the heart rate is 60 beats per minute, there is one heartbeat, or cycle, per second. So, the period is 1 second.

$$1 = \frac{2\pi}{|b|} \quad \text{Write an equation.}$$
$$|b| = 2\pi \quad \text{Multiply each side by } |b|.$$
$$b = \pm 2\pi \quad \text{Solve.}$$

For this example, let $b = 2\pi$. The use of the positive or negative value depends upon whether you begin a cycle with a maximum value (positive) or a minimum value (negative).

- There is no phase shift, so $h = 0$. So, the equation is $P = 20 \sin 2\pi t + 100$.

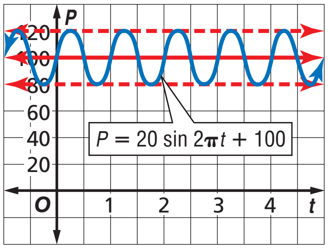
- Graph the function.

Step 1 Draw the midline $P = 100$.

Step 2 Draw maximum and minimum reference lines.

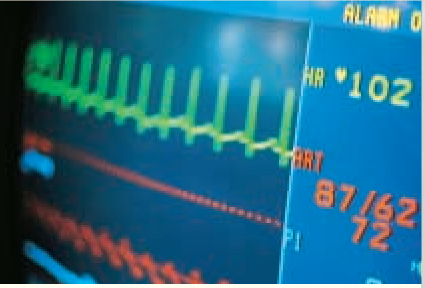
Step 3 Use the period to draw the graph of the function.

Step 4 There is no phase shift.



Examine Notice that each cycle begins at the midline, rises to 120, drops to 80, and then returns to the midline. This represents the blood pressure of 120 over 80 for one heartbeat. Since each cycle lasts 1 second, there will be 60 cycles, or heartbeats, in 1 minute. Therefore, the graph accurately represents the information.

More About. . .



Health

Blood pressure can change from minute to minute and can be affected by the slightest of movements, such as tapping your fingers or crossing your arms.

Source: American Heart Association

Check for Understanding

Concept Check

1. **Identify** the vertical shift, amplitude, period, and phase shift of the graph of $y = 3\cos(2x - 90°) + 15$.
2. **Define** the midline of a trigonometric graph.
3. **OPEN ENDED** Write the equation of a trigonometric function with a phase shift of $-45°$.

Guided Practice

State the amplitude, period, and phase shift for each function. Then graph the function.

4. $y = \sin\left(\theta - \frac{\pi}{2}\right)$
5. $y = \tan(\theta + 60°)$
6. $y = \cos(\theta - 45°)$
7. $y = \sec\left(\theta + \frac{\pi}{3}\right)$

State the vertical shift, equation of the midline, amplitude, and period for each function. Then graph the function.

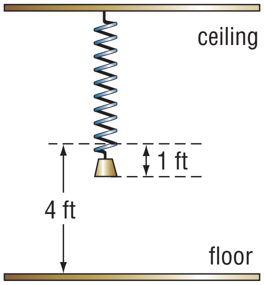
8. $y = \cos\theta + \frac{1}{4}$
9. $y = \sec\theta - 5$
10. $y = \tan\theta + 4$
11. $y = \sin\theta + 0.25$

State the vertical shift, amplitude, period, and phase shift of each function. Then graph the function.

12. $y = 3\sin[2(\theta - 30°)] + 10$
13. $y = 2\cot(3\theta + 135°) - 6$
14. $y = \frac{1}{2}\sec\left[4\left(\theta - \frac{\pi}{4}\right)\right] + 1$
15. $y = \frac{2}{3}\cos\left[\frac{1}{2}\left(\theta + \frac{\pi}{6}\right)\right] - 2$

Application

PHYSICS **For Exercises 16–18, use the following information.**
A weight is attached to a spring and suspended from the ceiling. At equilibrium, the weight is located 4 feet above the floor. The weight is pulled down 1 foot and released.



16. Determine the vertical shift, amplitude, and period of a function that represents the height of the weight above the floor if the weight returns to its lowest position every 4 seconds.
17. Write the equation for the height h of the weight above the floor as a function of time t seconds.
18. Draw a graph of the function you wrote in Exercise 17.

Practice and Apply

State the amplitude, period, and phase shift for each function. Then graph the function.

19. $y = \cos(\theta + 90°)$
20. $y = \cot(\theta - 30°)$
21. $y = \sin\left(\theta - \frac{\pi}{4}\right)$
22. $y = \cos\left(\theta + \frac{\pi}{3}\right)$
23. $y = \frac{1}{4}\tan(\theta + 22.5°)$
24. $y = 3\sin(\theta - 75°)$

Homework Help

For Exercises	See Examples
19–24	1
25–30	2
31, 32	1, 2
33–42	3
37–40	4

Extra Practice

See page 859.

State the vertical shift, equation of the midline, amplitude, and period for each function. Then graph the function.

25. $y = \sin \theta - 1$

26. $y = \sec \theta + 2$

27. $y = \cos \theta - 5$

28. $y = \csc \theta - \frac{3}{4}$

29. $y = \frac{1}{2} \sin \theta + \frac{1}{2}$

30. $y = 6 \cos \theta + 1.5$

31. Graph $y = 5 + \tan \left(\theta + \frac{\pi}{4}\right)$. Describe the transformation to the parent graph $y = \tan \theta$.

32. Draw a graph of the function $y = \frac{2}{3} \cos (\theta - 50°) + 2$. How does this graph compare to the graph of $y = \cos \theta$?

State the vertical shift, amplitude, period, and phase shift of each function. Then graph the function.

33. $y = 2 \sin [3(\theta - 45°)] + 1$

34. $y = 4 \cos [2(\theta + 30°)] - 5$

35. $y = 3 \csc \left[\frac{1}{2}(\theta + 60°)\right] - 3.5$

36. $y = 6 \cot \left[\frac{2}{3}(\theta - 90°)\right] + 0.75$

37. $y = \frac{1}{4} \cos (2\theta - 150°) + 1$

38. $y = \frac{2}{5} \tan (6\theta + 135°) - 4$

39. $y = 3 + 2 \sin \left[2\left(\theta + \frac{\pi}{4}\right)\right]$

40. $y = 4 + 5 \sec \left[\frac{1}{3}\left(\theta + \frac{2\pi}{3}\right)\right]$

41. Graph $y = 3 - \frac{1}{2} \cos \theta$ and $y = 3 + \frac{1}{2} \cos (\theta + \pi)$. How do the graphs compare?

42. Compare the graphs of $y = -\sin \left[\frac{1}{4}\left(\theta - \frac{\pi}{2}\right)\right]$ and $y = \cos \left[\frac{1}{4}\left(\theta + \frac{3\pi}{2}\right)\right]$.

WebQuest

Translations of trigonometric graphs can be used to describe temperature trends. Visit www.algebra2.com/webquest to continue work on your WebQuest project.

43. MUSIC When represented on an oscilloscope, the note A above middle C has period of $\frac{1}{440}$. Which of the following can be an equation for an oscilloscope graph of this note? The amplitude of the graph is K.

a. $y = K \sin 220\pi t$ **b.** $y = K \sin 440\pi t$ **c.** $y = K \sin 880\pi t$

ZOOLOGY **For Exercises 44–46, use the following information.**
The population of predators and prey in a closed ecological system tends to vary periodically over time. In a certain system, the population of owls O can be represented by $O = 150 + 30 \sin \left(\frac{\pi}{10}t\right)$ where t is the time in years since January 1, 2001. In that same system, the population of mice M can be represented by $M = 600 + 300 \sin \left(\frac{\pi}{10}t + \frac{\pi}{20}\right)$.

44. Find the maximum number of owls. After how many years does this occur?

45. What is the minimum number of mice? How long does it take for the population of mice to reach this level?

46. Why would the maximum owl population follow behind the population of mice?

47. TIDES The height of the water in a harbor rose to a maximum height of 15 feet at 6:00 P.M. and then dropped to a minimum level of 3 feet by 3:00 A.M. Assume that the water level can be modeled by the sine function. Write an equation that represents the height h of the water t hours after noon on the first day.

Online Research **Data Update** Use the Internet or another resource to find tide data for a location of your choice. Write a sine function to represent your data. Then graph the function. Visit www.algebra2.com/data_update to learn more.

www.algebra2.com/self_check_quiz

48. **CRITICAL THINKING** The graph of $y = \cot \theta$ is a transformation of the graph of $y = \tan \theta$. Determine a, b, and h so that $\cot \theta = a \tan [b(\theta - h)]$ for all values of θ for which each function is defined.

49. **WRITING IN MATH** Answer the question that was posed at the beginning of the lesson.

How can translations of trigonometric graphs be used to show animal populations?

Include the following in your answer:

- a description of what each number in the equation $R = 1200 + 250 \sin \frac{1}{2}\pi t$ represents, and
- a comparison of the graphs of $y = a \cos bx$, $y = a \cos bx + k$, and $y = a \cos [b(x - h)]$.

Standardized Test Practice

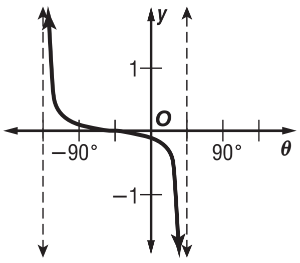
50. Which equation is represented by the graph?

Ⓐ $y = \cot (\theta + 45°)$

Ⓑ $y = \cot (\theta - 45°)$

Ⓒ $y = \tan (\theta + 45°)$

Ⓓ $y = \tan (\theta - 45°)$



51. Identify the equation for a sine function of period 90°, after a phase shift 20° to the left.

Ⓐ $y = \sin [0.25(\theta - 20°)]$ Ⓑ $y = \sin [4(\theta - 20°)]$

Ⓒ $y = \sin [0.25(\theta + 20°)]$ Ⓓ $y = \sin [4(\theta + 20°)]$

Maintain Your Skills

Mixed Review

Find the amplitude, if it exists, and period of each function. Then graph each function. *(Lesson 14-1)*

52. $y = 3 \csc \theta$ 53. $y = \sin \frac{\theta}{2}$ 54. $y = 3 \tan \frac{2}{3}\theta$

Find each value. *(Lesson 13-7)*

55. $\sin \left(\text{Cos}^{-1} \frac{2}{3}\right)$ 56. $\cos \left(\text{Cos}^{-1} \frac{4}{7}\right)$ 57. $\text{Sin}^{-1} \left(\sin \frac{5}{6}\right)$ 58. $\cos \left(\text{Tan}^{-1} \frac{3}{4}\right)$

59. **GEOMETRY** Find the total number of diagonals that can be drawn in a decagon. *(Lesson 12-2)*

Solve each equation. Round to the nearest hundredth. *(Lesson 10-4)*

60. $4^x = 24$ 61. $4.3^{3x+1} = 78.5$ 62. $7^{x-2} = 53^{-x}$

Simplify each expression. *(Lesson 9-4)*

63. $\frac{3}{a-2} + \frac{2}{a-3}$ 64. $\frac{w+12}{4w-16} - \frac{w+4}{2w-8}$ 65. $\frac{3y+1}{2y-10} + \frac{1}{y^2-2y-15}$

Getting Ready for the Next Lesson

PREREQUISITE SKILL **Find the value of each function.**
*(To review **reference angles**, see Lesson 13-3.)*

66. $\cos 150°$ 67. $\tan 135°$ 68. $\sin \frac{3\pi}{2}$ 69. $\cos \left(-\frac{\pi}{3}\right)$

70. $\sin (-\pi)$ 71. $\tan \left(-\frac{5\pi}{6}\right)$ 72. $\cos 225°$ 73. $\tan 405°$

CONTENTS

14-3 Trigonometric Identities

What You'll Learn

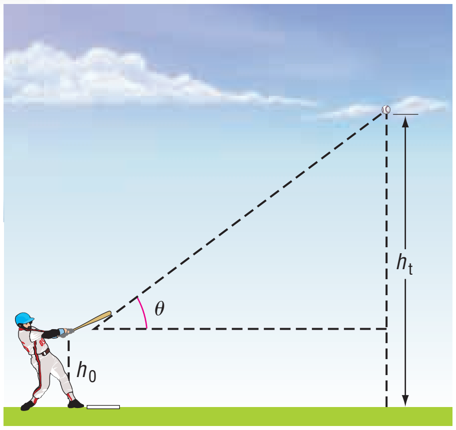
- Use identities to find trigonometric values.
- Use trigonometric identities to simplify expressions.

Vocabulary

- trigonometric identity

How can trigonometry be used to model the path of a baseball?

A model for the height of a baseball after it is hit as a function of time can be determined using trigonometry. If the ball is hit with an initial velocity of v feet per second at an angle of θ from the horizontal, then the height h of the ball after t seconds can be represented by $h = \left(\frac{-16}{v^2\cos^2\theta}\right)t^2 + \left(\frac{\sin\theta}{\cos\theta}\right)t + h_0$, where h_0 is the height of the ball in feet the moment it is hit.



FIND TRIGONOMETRIC VALUES In the equation above, the second term $\left(\frac{\sin\theta}{\cos\theta}\right)t$ can also be written as $(\tan\theta)t$. $\left(\frac{\sin\theta}{\cos\theta}\right)t = (\tan\theta)t$ is an example of a trigonometric identity. A **trigonometric identity** is an equation involving trigonometric functions that is true for all values for which every expression in the equation is defined.

The identity $\tan\theta = \frac{\sin\theta}{\cos\theta}$ is true except for angle measures such as 90°, 270°, 450°, …, $90° + 180° \cdot k$. The cosine of each of these angle measures is 0, so none of the expressions tan 90°, tan 270°, tan 450°, and so on, are defined. An identity similar to this is $\cot\theta = \frac{\cos\theta}{\sin\theta}$.

These identities are sometimes called *quotient identities*. These and other basic trigonometric identities are listed below.

Key Concept — Basic Trigonometric Identities

Quotient Identities	$\tan\theta = \frac{\sin\theta}{\cos\theta}$	$\cot\theta = \frac{\cos\theta}{\sin\theta}$	
Reciprocal Identities	$\csc\theta = \frac{1}{\sin\theta}$	$\sec\theta = \frac{1}{\cos\theta}$	$\cot\theta = \frac{1}{\tan\theta}$
Pythagorean Identities	$\cos^2\theta + \sin^2\theta = 1$ $\tan^2\theta + 1 = \sec^2\theta$ $\cot^2\theta + 1 = \csc^2\theta$		

You can use trigonometric identities to find values of trigonometric functions.

Example 1 Find a Value of a Trigonometric Function

a. Find $\cos\theta$ if $\sin\theta = -\frac{3}{5}$ and $90° < \theta < 180°$.

$\cos^2\theta + \sin^2\theta = 1$ Trigonometric identity

$\cos^2\theta = 1 - \sin^2\theta$ Subtract $\sin^2\theta$ from each side.

$\cos^2\theta = 1 - \left(\frac{3}{5}\right)^2$ Substitute $\frac{3}{5}$ for $\sin\theta$.

$\cos^2\theta = 1 - \frac{9}{25}$ Square $\frac{3}{5}$.

$\cos^2\theta = \frac{16}{25}$ Subtract.

$\cos\theta = \pm\frac{4}{5}$ Take the square root of each side.

Since θ is in the second quadrant, $\cos\theta$ is negative. Thus, $\cos\theta = -\frac{4}{5}$.

b. Find $\csc\theta$ if $\cot\theta = -\frac{1}{4}$ and $270° < \theta < 360°$.

$\cot^2\theta + 1 = \csc^2\theta$ Trigonometric identity

$\left(-\frac{1}{4}\right)^2 + 1 = \csc^2\theta$ Substitute $-\frac{1}{4}$ for $\cot\theta$.

$\frac{1}{16} + 1 = \csc^2\theta$ Square $-\frac{1}{4}$.

$\frac{17}{16} = \csc^2\theta$ Add.

$\pm\frac{\sqrt{17}}{4} = \csc\theta$ Take the square root of each side.

Since θ is in the fourth quadrant, $\csc\theta$ is negative. Thus, $\csc\theta = -\frac{\sqrt{17}}{4}$.

SIMPLIFY EXPRESSIONS Trigonometric identities can also be used to simplify expressions containing trigonometric functions. Simplifying an expression that contains trigonometric functions means that the expression is written as a numerical value or in terms of a single trigonometric function, if possible.

Example 2 Simplify an Expression

Simplify $\frac{\csc^2\theta - \cot^2\theta}{\cos\theta}$.

$\frac{\csc^2\theta - \cot^2\theta}{\cos\theta} = \frac{\frac{1}{\sin^2\theta} - \frac{\cos^2\theta}{\sin^2\theta}}{\cos\theta}$ $\csc^2\theta = \frac{1}{\sin^2\theta}$, $\cot^2\theta = \frac{\cos^2\theta}{\sin^2\theta}$

$= \frac{\frac{1 - \cos^2\theta}{\sin^2\theta}}{\cos\theta}$ Add.

$= \frac{\frac{\sin^2\theta}{\sin^2\theta}}{\cos\theta}$ $1 - \cos^2\theta = \sin^2\theta$

$= \frac{1}{\cos\theta}$ $\frac{\sin^2\theta}{\sin^2\theta} = 1$

$= \sec\theta$ $\frac{1}{\cos\theta} = \sec\theta$

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Example 3 Simplify and Use an Expression

BASEBALL **Refer to the application at the beginning of the lesson. Rewrite the equation in terms of tan θ.**

$$h = \left(\frac{-16}{v^2\cos^2\theta}\right)t^2 + \left(\frac{\sin\theta}{\cos\theta}\right)t + h_0$$ Original equation

$$= \frac{-16}{v^2}\left(\frac{1}{\cos^2\theta}\right)t^2 + \left(\frac{\sin\theta}{\cos\theta}\right)t + h_0$$ Factor.

$$= \frac{-16}{v^2}\left(\frac{1}{\cos^2\theta}\right)t^2 + (\tan\theta)t + h_0$$ $\frac{\sin\theta}{\cos\theta} = \tan\theta$

$$= \frac{-16}{v^2}(\sec^2\theta)t^2 + (\tan\theta)t + h_0$$ Since $\frac{1}{\cos\theta} = \sec\theta$, $\frac{1}{\cos^2\theta} = \sec^2\theta$.

$$= \frac{-16}{v^2}(1 + \tan^2\theta)t^2 + (\tan\theta)t + h_0$$ $\sec^2\theta = 1 + \tan^2\theta$

Thus, $\left(\frac{-16}{v^2\cos^2\theta}\right)t^2 + \left(\frac{\sin\theta}{\cos\theta}\right)t + h_0 = \frac{-16}{v^2}(1 + \tan^2\theta)t^2 + (\tan\theta)t + h_0$.

Check for Understanding

Concept Check

1. **Describe** how you can determine the quadrant in which the terminal side of angle α lies if $\sin\alpha = -\frac{1}{4}$.
2. **Explain** why the Pythagorean identities are so named.
3. **OPEN ENDED** Explain what it means to simplify a trigonometric expression.

Guided Practice

Find the value of each expression.

4. $\tan\theta$, if $\sin\theta = \frac{1}{2}$; $90° \leq \theta < 180°$
5. $\csc\theta$, if $\cos\theta = -\frac{3}{5}$; $180° \leq \theta < 270°$
6. $\cos\theta$, if $\sin\theta = \frac{4}{5}$; $0° \leq \theta < 90°$
7. $\sec\theta$, if $\tan\theta = -1$; $270° < \theta < 360°$

Simplify each expression.

8. $\csc\theta\cos\theta\tan\theta$
9. $\sec^2\theta - 1$
10. $\frac{\tan\theta}{\sin\theta}$
11. $\sin\theta\,(1 + \cot^2\theta)$

Application

12. **PHYSICAL SCIENCE** When a person moves along a circular path, the body leans away from a vertical position. The nonnegative acute angle that the body makes with the vertical is called the *angle of inclination* and is represented by the equation $\tan\theta = \frac{v^2}{gR}$, where R is the radius of the circular path, v is the speed of the person in meters per second, and g is the acceleration due to gravity, 9.8 meters per second squared. Write an equivalent expression using $\sin\theta$ and $\cos\theta$.

Practice and Apply

Practice and Apply

Find the value of each expression.

13. $\tan\theta$, if $\cot\theta = 2$; $0° \leq \theta < 90°$
14. $\sin\theta$, if $\cos\theta = \frac{2}{3}$; $0° \leq \theta < 90°$
15. $\sec\theta$, if $\tan\theta = -2$; $90° < \theta < 180°$
16. $\tan\theta$, if $\sec\theta = -3$; $180° < \theta < 270°$

Homework Help

For Exercises	See Examples
13–24	1
25–36	2
37–43	3

Extra Practice

See page 860.

Find the value of each expression.

17. $\csc \theta$, if $\cos \theta = -\frac{3}{5}$; $90° < \theta < 180°$

18. $\cos \theta$, if $\sec \theta = \frac{5}{3}$; $270° < \theta < 360°$

19. $\cos \theta$, if $\sin \theta = \frac{1}{2}$; $0° \leq \theta < 90°$

20. $\csc \theta$, if $\cos \theta = -\frac{2}{3}$; $180° < \theta < 270°$

21. $\tan \theta$, if $\cos \theta = \frac{4}{5}$; $0° \leq \theta < 90°$

22. $\cos \theta$, if $\csc \theta = -\frac{5}{3}$; $270° < \theta < 360°$

23. $\sec \theta$, if $\sin \theta = \frac{3}{4}$; $90° < \theta < 180°$

24. $\sin \theta$, if $\tan \theta = 4$; $180° < \theta < 270°$

Simplify each expression.

25. $\cos \theta \csc \theta$

26. $\tan \theta \cot \theta$

27. $\sin \theta \cot \theta$

28. $\cos \theta \tan \theta$

29. $2(\csc^2 \theta - \cot^2 \theta)$

30. $3(\tan^2 \theta - \sec^2 \theta)$

31. $\frac{\cos \theta \csc \theta}{\tan \theta}$

32. $\frac{\sin \theta \csc \theta}{\cot \theta}$

33. $\frac{1 - \cos^2 \theta}{\sin^2 \theta}$

34. $\frac{1 - \sin^2 \theta}{\sin^2 \theta}$

35. $\frac{\sin^2 \theta + \cos^2 \theta}{\sin^2 \theta}$

36. $\frac{\tan^2 \theta - \sin^2 \theta}{\tan^2 \theta \sin^2 \theta}$

AMUSEMENT PARKS **For Exercises 37–39, use the following information.**
Suppose a child is riding on a merry-go-round and is seated on an outside horse. The diameter of the merry-go-round is 16 meters.

37. If the sine of the angle of inclination of the child is $\frac{1}{5}$, what is the angle of inclination made by the child? Refer to Exercise 12 for information on the angle of inclination.

38. What is the velocity of the merry-go-round?

39. If the speed of the merry-go-round is 3.6 meters per second, what is the value of the angle of inclination of a rider?

More About. . .

Amusement Parks

The oldest operational carousel in the United States is the Flying Horse Carousel at Martha's Vineyard, Massachusetts.

Source: Martha's Vineyard Preservation Trust

LIGHTING **For Exercises 40 and 41, use the following information.**
The amount of light that a source provides to a surface is called the illuminance. The illuminance E in foot candles on a surface is related to the distance R in feet from the light source. The formula $\sec \theta = \frac{I}{ER^2}$, where I is the intensity of the light source measured in candles and θ is the angle between the light beam and a line perpendicular to the surface, can be used in situations in which lighting is important.

40. Solve the formula in terms of E.

41. Is the equation in Exercise 40 equivalent to $R^2 = \frac{I \tan \theta \cos \theta}{E}$? Explain.

ELECTRONICS **For Exercises 42 and 43, use the following information.**
When an alternating current of frequency f and a peak current I pass through a resistance R, then the power delivered to the resistance at time t seconds is $P = I^2R - I^2R \cos^2 (2ft\pi)$.

42. Write an expression for the power in terms of $\sin^2 2ft\pi$.

43. Write an expression for the power in terms of $\tan^2 2ft\pi$.

44. **CRITICAL THINKING** If $\tan \beta = \frac{3}{4}$, find $\frac{\sin \beta \sec \beta}{\cot \beta}$.

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45. **WRITING IN MATH** Answer the question that was posed at the beginning of the lesson.

How can trigonometry be used to model the path of a baseball?

Include the following in your answer:

- an explanation of why the equation at the beginning of the lesson is the same as $y = \frac{-16 \sec^2 \theta}{v^2}x^2 + (\tan \theta)x + h_0$, and
- examples of how you might use this equation for other situations.

Standardized Test Practice

46. If $\sin x = m$ and $0 < x < 90°$, then $\tan x =$

Ⓐ $\frac{1}{m^2}$. Ⓑ $\frac{m}{\sqrt{1 - m^2}}$. Ⓒ $\frac{1 - m^2}{m}$. Ⓓ $\frac{m}{1 - m^2}$.

47. $\frac{1}{1 + \sin x} + \frac{1}{1 - \sin x} =$

Ⓐ $2 \sec^2 x$ Ⓑ $-\sec^2 x$ Ⓒ $2 \csc^2 x$ Ⓓ $-\csc^2 x$

Maintain Your Skills

Mixed Review

State the vertical shift, equation of the midline, amplitude, and period for each function. Then graph the function. *(Lesson 14-2)*

48. $y = \sin \theta - 1$

49. $y = \tan \theta + 12$

Find the amplitude, if it exists, and period of each function. Then graph each function. *(Lesson 14-1)*

50. $y = \csc 2\theta$

51. $y = \cos 3\theta$

52. $y = \frac{1}{3} \cot 5\theta$

53. Find the sum of a geometric series for which $a_1 = 48$, $a_n = 3$, and $r = \frac{1}{2}$. *(Lesson 11-4)*

54. Write an equation of a parabola with focus at $(11, -1)$ and whose directrix is $y = 2$. *(Lesson 8-2)*

Getting Ready for the Next Lesson

PREREQUISITE SKILL **Name the property illustrated by each statement.**
*(To review **properties of equality**, see Lesson 1-3.)*

55. If $4 + 8 = 12$, then $12 = 4 + 8$.

56. If $7 + s = 21$, then $s = 14$.

57. If $4x = 16$, then $12x = 48$.

58. If $q + (8 + 5) = 32$, then $q + 13 = 32$.

Practice Quiz 1

Lessons 14-1 through 14-3

1. Find the amplitude and period of $y = \frac{3}{4} \sin \frac{1}{2}\theta$. Then graph the function. *(Lesson 14-1)*

2. State the vertical shift, amplitude, period, and phase shift for $y = 2 \cos \left[\frac{1}{4}\left(\theta - \frac{\pi}{4}\right)\right] - 5$. Then graph the function. *(Lesson 14-2)*

Find the value of each expression. *(Lesson 14-3)*

3. $\cos \theta$, if $\sin \theta = \frac{4}{5}$; $90° < \theta < 180°$

4. $\csc \theta$, if $\cot \theta = -\frac{2}{3}$; $270° < \theta < 360°$

5. $\sec \theta$, if $\tan \theta = \frac{1}{2}$; $0° < \theta < 90°$

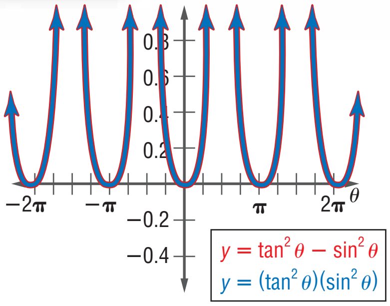
14-4 Verifying Trigonometric Identities

What You'll Learn

- Verify trigonometric identities by transforming one side of an equation into the form of the other side.
- Verify trigonometric identities by transforming each side of the equation into the same form.

How can you verify trigonometric identities?

Examine the graphs of $y = \tan^2 \theta - \sin^2 \theta$ and $y = \tan^2 \theta \sin^2 \theta$. Recall that when the graphs of two functions coincide, the functions are equivalent. However, the graphs only show a limited range of solutions. It is not sufficient to show some values of θ and conclude that the statement is true for all values of θ. In order to show that the equation $\tan^2 \theta - \sin^2 \theta = \tan^2 \theta \sin^2 \theta$ for all values of θ, you must consider the general case.



Study Tip

Common Misconception

You cannot perform operations to the quantities from each side of an unverified identity as you do with equations. Until an identity is verified it is not considered an equation, so the properties of equality do not apply.

TRANSFORM ONE SIDE OF AN EQUATION You can use the basic trigonometric identities along with the definitions of the trigonometric functions to verify identities. For example, if you wish to show that $\tan^2 \theta - \sin^2 \theta = \tan^2 \theta \sin^2 \theta$ is an identity, you need to show that it is true for all values of θ.

Verifying an identity is like checking the solution of an equation. You must simplify one or both sides of an equation *separately* until they are the same. In many cases, it is easier to work with only one side of an equation. You may choose either side, but it is often easier to begin with the more complicated side of the equation. Transform that expression into the form of the simpler side.

Example 1 Transform One Side of an Equation

Verify that $\tan^2 \theta - \sin^2 \theta = \tan^2 \theta \sin^2 \theta$ is an identity.

Transform the left side.

$$\tan^2 \theta - \sin^2 \theta \stackrel{?}{=} \tan^2 \theta \sin^2 \theta \quad \text{Original equation}$$

$$\frac{\sin^2 \theta}{\cos^2 \theta} - \sin^2 \theta \stackrel{?}{=} \tan^2 \theta \sin^2 \theta \quad \tan^2 \theta = \frac{\sin^2 \theta}{\cos^2 \theta}$$

$$\frac{\sin^2 \theta}{\cos^2 \theta} - \frac{\sin^2 \theta \cos^2 \theta}{\cos^2 \theta} \stackrel{?}{=} \tan^2 \theta \sin^2 \theta \quad \text{Rewrite using the LCD, } \cos^2 \theta.$$

$$\frac{\sin^2 \theta - \sin^2 \theta \cos^2 \theta}{\cos^2 \theta} \stackrel{?}{=} \tan^2 \theta \sin^2 \theta \quad \text{Subtract.}$$

$$\frac{\sin^2 \theta\,(1 - \cos^2 \theta)}{\cos^2 \theta} \stackrel{?}{=} \tan^2 \theta \sin^2 \theta \quad \text{Factor.}$$

$$\frac{\sin^2 \theta \sin^2 \theta}{\cos^2 \theta} \stackrel{?}{=} \tan^2 \theta \sin^2 \theta \quad 1 - \cos^2 \theta = \sin^2 \theta$$

$$\frac{\sin^2 \theta}{\cos^2 \theta} \cdot \frac{\sin^2 \theta}{1} \stackrel{?}{=} \tan^2 \theta \sin^2 \theta \quad \frac{ab}{c} = \frac{a}{c} \cdot \frac{b}{1}$$

$$\tan^2 \theta \sin^2 \theta = \tan^2 \theta \sin^2 \theta \quad \frac{\sin^2 \theta}{\cos^2 \theta} = \tan \theta$$

CONTENTS

Standardized Test Practice

A B C D

Example 2 Find an Equivalent Expression

Multiple-Choice Test Item

$\sin\theta\left(\dfrac{1}{\sin\theta} - \dfrac{\cos\theta}{\cot\theta}\right) =$

Ⓐ $\cos\theta$ Ⓑ $\sin\theta$ Ⓒ $\cos^2\theta$ Ⓓ $\sin^2\theta$

Read the Test Item

Find an expression that is equal to the given expression.

Solve the Test Item

Write a trigonometric identity by using the basic trigonometric identities and the definitions of trigonometric functions to transform the given expression to match one of the choices.

$$\begin{aligned}
\sin\theta\left(\frac{1}{\sin\theta} - \frac{\cos\theta}{\cot\theta}\right) &= \sin\theta\left(\frac{1}{\sin\theta} - \frac{\cos\theta}{\frac{\cos\theta}{\sin\theta}}\right) && \cot\theta = \frac{\cos\theta}{\sin\theta}\\
&= \sin\theta\left(\frac{1}{\sin\theta} - \frac{\cos\theta\sin\theta}{\cos\theta}\right) && \text{Simplify.}\\
&= \sin\theta\left(\frac{1}{\sin\theta} - \sin\theta\right) && \text{Simplify.}\\
&= 1 - \sin^2\theta && \text{Distributive property.}\\
&= \cos^2\theta && 1 - \sin^2\theta = \cos^2\theta
\end{aligned}$$

Since $\sin\theta\left(\dfrac{1}{\sin\theta} - \dfrac{\cos\theta}{\cot\theta}\right) = \cos^2\theta$, the answer is C.

The Princeton Review

Test-Taking Tip

Verify your answer by choosing values for θ. Then evaluate the original expression and compare to your answer choice.

TRANSFORM BOTH SIDES OF AN EQUATION Sometimes it is easier to transform both sides of an equation separately into a common form. The following suggestions may be helpful as you verify trigonometric identities.

- Substitute one or more basic trigonometric identities to simplify an expression.
- Factor or multiply to simplify an expression.
- Multiply both the numerator and denominator by the same trigonometric expression.
- Write both sides of the identity in terms of sine and cosine only. Then simplify each side as much as possible.

Example 3 Verify by Transforming Both Sides

Verify that $\sec^2\theta - \tan^2\theta = \tan\theta\cot\theta$ is an identity.

$$\begin{aligned}
\sec^2\theta - \tan^2\theta &\stackrel{?}{=} \tan\theta\cot\theta && \text{Original equation}\\
\frac{1}{\cos^2\theta} - \frac{\sin^2\theta}{\cos^2\theta} &\stackrel{?}{=} \frac{\sin\theta}{\cos\theta}\cdot\frac{\cos\theta}{\sin\theta} && \text{Express all terms using sine and cosine.}\\
\frac{1 - \sin^2\theta}{\cos^2\theta} &\stackrel{?}{=} 1 && \text{Subtract on the left. Multiply on the right.}\\
\frac{\cos^2\theta}{\cos^2\theta} &\stackrel{?}{=} 1 && 1 - \sin^2\theta = \cos^2\theta\\
1 &= 1 && \text{Simplify the left side.}
\end{aligned}$$

www.algebra2.com/extra_examples

Check for Understanding

Concept Check

1. **Explain** the steps used to verify the identity $\sin\theta \tan\theta = \sec\theta - \cos\theta$.
2. **Describe** the various methods you can use to show that two trigonometric expressions form an identity.
3. **OPEN ENDED** Write a trigonometric equation that is not an identity. Explain how you know it is not an identity.

Guided Practice

Verify that each of the following is an identity.

4. $\tan\theta(\cot\theta + \tan\theta) = \sec^2\theta$
5. $\tan^2\theta\cos^2\theta = 1 - \cos^2\theta$
6. $\dfrac{\cos^2\theta}{1 - \sin\theta} = 1 + \sin\theta$
7. $\dfrac{1 + \tan^2\theta}{\csc^2\theta} = \tan^2\theta$
8. $\dfrac{\sin\theta}{\sec\theta} = \dfrac{1}{\tan\theta + \cot\theta}$
9. $\dfrac{\sec\theta + 1}{\tan\theta} = \dfrac{\tan\theta}{\sec\theta - 1}$

Standardized Test Practice

10. Which expression is equivalent to $\dfrac{\sec\theta + \csc\theta}{1 + \tan\theta}$?

Ⓐ $\sin\theta$ Ⓑ $\cos\theta$ Ⓒ $\tan\theta$ Ⓓ $\csc\theta$

Practice and Apply

Homework Help

For Exercises	See Examples
11–24, 26–32	1, 2
25	3

Extra Practice

See page 860.

Verify that each of the following is an identity.

11. $\cos^2\theta + \tan^2\theta\cos^2\theta = 1$
12. $\cot\theta(\cot\theta + \tan\theta) = \csc^2\theta$
13. $1 + \sec^2\theta\sin^2\theta = \sec^2\theta$
14. $\sin\theta\sec\theta\cot\theta = 1$
15. $\dfrac{1 - \cos\theta}{1 + \cos\theta} = (\csc\theta - \cot\theta)^2$
16. $\dfrac{1 - 2\cos^2\theta}{\sin\theta\cos\theta} = \tan\theta - \cot\theta$
17. $\cot\theta\csc\theta = \dfrac{\cot\theta + \csc\theta}{\sin\theta + \tan\theta}$
18. $\sin\theta + \cos\theta = \dfrac{1 + \tan\theta}{\sec\theta}$
19. $\dfrac{\sec\theta}{\sin\theta} - \dfrac{\sin\theta}{\cos\theta} = \cot\theta$
20. $\dfrac{\sin\theta}{1 - \cos\theta} + \dfrac{1 - \cos\theta}{\sin\theta} = 2\csc\theta$
21. $\dfrac{1 + \sin\theta}{\sin\theta} = \dfrac{\cot^2\theta}{\csc\theta - 1}$
22. $\dfrac{1 + \tan\theta}{1 + \cot\theta} = \dfrac{\sin\theta}{\cos\theta}$
23. $\dfrac{1}{\sec^2\theta} + \dfrac{1}{\csc^2\theta} = 1$
24. $1 + \dfrac{1}{\cos\theta} = \dfrac{\tan^2\theta}{\sec\theta - 1}$
25. $1 - \tan^4\theta = 2\sec^2\theta - \sec^4\theta$
26. $\cos^4\theta - \sin^4\theta = \cos^2\theta - \sin^2\theta$
27. $\dfrac{1 - \cos\theta}{\sin\theta} = \dfrac{\sin\theta}{1 + \cos\theta}$
28. $\dfrac{\cos\theta}{1 + \sin\theta} + \dfrac{\cos\theta}{1 - \sin\theta} = 2\sec\theta$
29. Verify that $\tan\theta\sin\theta\cos\theta\csc^2\theta = 1$ is an identity.
30. Show that $1 + \cos\theta$ and $\dfrac{\sin^2\theta}{1 - \cos\theta}$ form an identity.

PHYSICS **For Exercises 31 and 32, use the following information.**
If an object is propelled from ground level, the maximum height that it reaches is given by $h = \dfrac{v^2\sin^2\theta}{2g}$, where θ is the angle between the ground and the initial path of the object, v is the object's initial velocity, and g is the acceleration due to gravity, 9.8 meters per second squared.

31. Verify the identity $\dfrac{v^2\sin^2\theta}{2g} = \dfrac{v^2\tan^2\theta}{2g\sec^2\theta}$.
32. A model rocket is launched with an initial velocity of 110 meters per second at an angle of 80° with the ground. Find the maximum height of the rocket.

33. **CRITICAL THINKING** Present a logical argument for why the identity $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$ is true when $0 \leq x \leq 1$.

34. **WRITING IN MATH** Answer the question that was posed at the beginning of the lesson.

How can you verify trigonometric identities?

Include the following in your answer:

- an explanation of why you cannot perform operations to each side of an unverified identity,
- an explanation of how you can tell if two expressions are equivalent, and
- an explanation of why you cannot use the graphs of two equations to verify an identity.

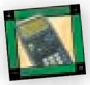
Standardized Test Practice

35. Which of the following is not equivalent to $\cos \theta$?

Ⓐ $\frac{\cos \theta}{\cos^2 \theta + \sin^2 \theta}$ Ⓑ $\frac{1 - \sin^2 \theta}{\cos \theta}$ Ⓒ $\cot \theta \sin \theta$ Ⓓ $\tan \theta \csc \theta$

36. Which of the following is equivalent to $\sin \theta + \cot \theta \cos \theta$?

Ⓐ $2 \sin \theta$ Ⓑ $\frac{1}{\sin \theta}$ Ⓒ $\cos^2 \theta$ Ⓓ $\frac{\sin \theta + \cos \theta}{\sin^2 \theta}$



Graphing Calculator

VERIFYING TRIGONOMETRIC IDENTITIES You can determine whether or not an equation may be a trigonometric identity by graphing the expressions on either side of the equals sign as two separate functions. If the graphs do not match, then the equation is not an identity. If the two graphs do coincide, the equation *might* be an identity. The equation has to be verified algebraically to ensure that it is an identity.

Determine whether each of the following *may be* or *is not* an identity.

37. $\cot x + \tan x = \csc x \cot x$

38. $\sec^2 x - 1 = \sin^2 x \sec^2 x$

39. $(1 + \sin x)(1 - \sin x) = \cos^2 x$

40. $\frac{1}{\sec x \tan x} = \csc x - \sin x$

41. $\frac{\sec^2 x}{\tan x} = \sec x \csc x$

42. $\frac{1}{\sec x} + \frac{1}{\csc x} = 1$

Maintain Your Skills

Mixed Review

Find the value of each expression. *(Lesson 14-3)*

43. $\sec \theta$, if $\tan \theta = \frac{1}{2}$; $0° < \theta < 90°$

44. $\cos \theta$, if $\sin \theta = -\frac{2}{3}$; $180° < \theta < 270°$

45. $\csc \theta$, if $\cot \theta = -\frac{7}{12}$; $90° < \theta < 180°$

46. $\sin \theta$, if $\cos \theta = \frac{3}{4}$; $270° < \theta < 360°$

State the amplitude, period, and phase shift of each function. Then graph each function. *(Lesson 14-2)*

47. $y = \cos(\theta - 30°)$

48. $y = \sin(\theta - 45°)$

49. $y = 3 \cos\left(\theta + \frac{\pi}{2}\right)$

50. What is the probability that an event occurs if the odds of the event occurring are 5:1? *(Lesson 12-4)*

Getting Ready for the Next Lesson

PREREQUISITE SKILL **Simplify each expression.**

*(To review **simplifying radical expressions**, see Lesson 5-6.)*

51. $\frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2}$

52. $\frac{1}{2} \cdot \frac{\sqrt{2}}{2}$

53. $\frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{2}$

54. $\frac{1}{2} - \frac{\sqrt{3}}{4}$

CONTENTS

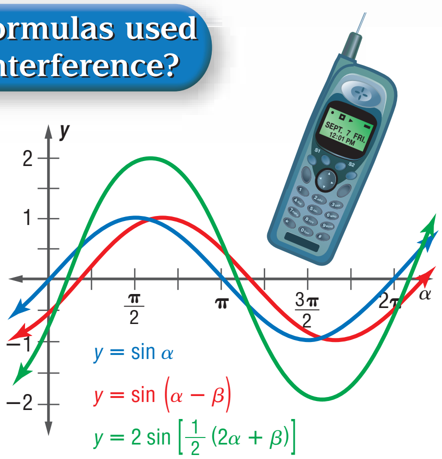
14-5 Sum and Difference of Angles Formulas

What You'll Learn

- Find values of sine and cosine involving sum and difference formulas.
- Verify identities by using sum and difference formulas.

How are the sum and difference formulas used to describe communication interference?

Have you ever been talking on a cell phone and temporarily lost the signal? Radio waves that pass through the same place at the same time cause interference. *Constructive interference* occurs when two waves combine to have a greater amplitude than either of the component waves. *Destructive interference* occurs when the component waves combine to have a smaller amplitude.



SUM AND DIFFERENCE FORMULAS Notice that the third equation shown above involves the sum of α and β. It is often helpful to use formulas for the trigonometric values of the difference or sum of two angles. For example, you could find sin 15° by evaluating sin (60° − 45°). Formulas can be developed that can be used to evaluate expressions like $\sin (\alpha - \beta)$ or $\cos (\alpha + \beta)$.

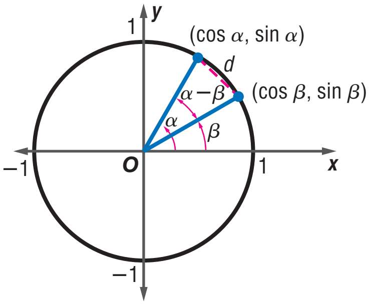
Study Tips

Reading Math

The Greek letter *beta*, β, can be used to denote the measure of an angle.

It is important to realize that $\sin (\alpha \pm \beta)$ is not the same as $\sin \alpha \pm \sin \beta$.

The figure at the right shows two angles α and β in standard position on the unit circle. Use the Distance Formula to find d, where $(x_1, y_1) = (\cos \beta, \sin \beta)$ and $(x_2, y_2) = (\cos \alpha, \sin \alpha)$.



$$d = \sqrt{(\cos \alpha - \cos \beta)^2 + (\sin \alpha - \sin \beta)^2}$$

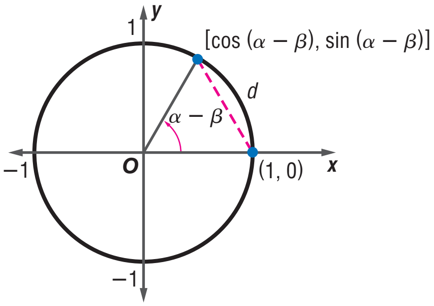
$$d^2 = (\cos \alpha - \cos \beta)^2 + (\sin \alpha - \sin \beta)^2$$

$$d^2 = (\cos^2 \alpha - 2\cos \alpha \cos \beta + \cos^2 \beta) + (\sin^2 \alpha - 2\sin \alpha \sin \beta + \sin^2 \beta)$$

$$d^2 = \cos^2 \alpha + \sin^2 \alpha + \cos^2 \beta + \sin^2 \beta - 2 \cos \alpha \cos \beta - 2 \sin \alpha \sin \beta$$

$$d^2 = 1 + 1 - 2 \cos \alpha \cos \beta - 2 \sin \alpha \sin \beta \qquad \sin^2 \alpha + \cos^2 \alpha = 1 \text{ and}$$

$$d^2 = 2 - 2 \cos \alpha \cos \beta - 2 \sin \alpha \sin \beta \qquad \sin^2 \beta + \cos^2 \beta = 1$$



Now find the value of d^2 when the angle having measure $\alpha - \beta$ is in standard position on the unit circle, as shown in the figure at the left.

$$d = \sqrt{[\cos (\alpha - \beta) - 1]^2 + [\sin (\alpha - \beta) - 0]^2}$$

$$\begin{aligned} d^2 &= [\cos (\alpha - \beta) - 1]^2 + [\sin (\alpha - \beta) - 0]^2 \\ &= [\cos^2 (\alpha - \beta) - 2 \cos (\alpha - \beta) + 1] + \sin^2 (\alpha - \beta) \\ &= \cos^2 (\alpha - \beta) + \sin^2 (\alpha - \beta) - 2 \cos (\alpha - \beta) + 1 \\ &= 1 - 2 \cos (\alpha - \beta) + 1 \\ &= 2 - 2 \cos (\alpha - \beta) \end{aligned}$$

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By equating the two expressions for d^2, you can find a formula for $\cos(\alpha - \beta)$.

$$\begin{aligned} d^2 &= d^2 \\ 2 - 2\cos(\alpha - \beta) &= 2 - 2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta \\ -1 + \cos(\alpha - \beta) &= -1 + \cos\alpha\cos\beta + \sin\alpha\sin\beta && \text{Divide each side by } -2. \\ \cos(\alpha - \beta) &= \cos\alpha\cos\beta + \sin\alpha\sin\beta && \text{Add 1 to each side.} \end{aligned}$$

Use the formula for $\cos(\alpha - \beta)$ to find a formula for $\cos(\alpha + \beta)$.

$$\begin{aligned} \cos(\alpha - \beta) &= \cos[\alpha - (-\beta)] \\ &= \cos\alpha\cos(-\beta) + \sin\alpha\sin(-\beta) \\ &= \cos\alpha\cos\beta - \sin\alpha\sin\beta && \cos(-\beta) = \cos\beta;\ \sin(-\beta) = -\sin\beta \end{aligned}$$

You can use a similar method to find formulas for $\sin(\alpha + \beta)$ and $\sin(\alpha - \beta)$.

Key Concept — Sum and Difference of Angles Formulas

The following identities hold true for all values of α and β.

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$

Notice the symbol ∓ in the formula for $\cos(\alpha \pm \beta)$. It means "minus or plus." In the cosine formula, when the sign on the left side of the equation is plus, the sign on the right side is minus; when the sign on the left side is minus, the sign on the right side is plus. The signs match each other in the sine formula.

Example 1 Use Sum and Difference of Angles Formulas

Find the exact value of each expression.

a. $\cos 75°$

Use the formula $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$.

$$\begin{aligned} \cos 75° &= \cos(30° + 45°) && \alpha = 30°, \beta = 45° \\ &= \cos 30°\cos 45° - \sin 30°\sin 45° \\ &= \left(\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2}\right) - \left(\frac{1}{2}\cdot\frac{\sqrt{2}}{2}\right) && \text{Evaluate each expression.} \\ &= \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4} && \text{Multiply.} \\ &= \frac{\sqrt{6} - \sqrt{2}}{4} && \text{Simplify.} \end{aligned}$$

b. $\sin(-210°)$

Use the formula $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$.

$$\begin{aligned} \sin(-210°) &= \sin(60° - 270°) && \alpha = 60°, \beta = 270° \\ &= \sin 60°\cos 270° - \cos 60°\sin 270° \\ &= \left(\frac{\sqrt{3}}{2}\right)(0) - \left(\frac{1}{2}\right)(-1) && \text{Evaluate each expression.} \\ &= 0 - \left(-\frac{1}{2}\right) && \text{Multiply.} \\ &= \frac{1}{2} && \text{Simplify.} \end{aligned}$$

www.algebra2.com/extra_examples

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Example 2 Use Sum and Difference Formulas to Solve a Problem

Study Tip

Reading Math
The symbol ϕ is the lowercase Greek letter *phi*.

PHYSICS **On June 22, the maximum amount of light energy falling on a square foot of ground at a location in the northern hemisphere is given by $E \sin (113.5° - \phi)$, where ϕ is the latitude of the location and E is the amount of light energy when the Sun is directly overhead. Use the difference of angles formula to determine the amount of light energy in Rochester, New York, located at a latitude of 43.1° N.**

Use the difference formula for sine.

$$\begin{aligned}\sin (113.5° - \phi) &= \sin 113.5° \cos \phi - \cos 113.5° \sin \phi \\ &= \sin 113.5° \cos 43.1° - \cos 113.5° \sin 43.1° \\ &= 0.9171 \cdot 0.7301 - (-0.3987) \cdot 0.6833 \\ &= 0.9420\end{aligned}$$

In Rochester, New York, the maximum light energy per square foot is $0.9420E$.

VERIFY IDENTITIES You can also use the sum and difference formulas to verify identities.

Example 3 Verify Identities

Verify that each of the following is an identity.

a. $\sin (180° + \theta) = -\sin \theta$

$$\begin{aligned}\sin (180° + \theta) &\stackrel{?}{=} -\sin \theta && \text{Original equation} \\ \sin 180° \cos \theta + \cos 180° \sin \theta &\stackrel{?}{=} -\sin \theta && \text{Sum of angles formula} \\ 0 \cos \theta + (-1) \sin \theta &\stackrel{?}{=} -\sin \theta && \text{Evaluate each expression.} \\ -\sin \theta &= -\sin \theta && \text{Simplify.}\end{aligned}$$

b. $\cos (180° + \theta) = -\cos \theta$

$$\begin{aligned}\cos (180° + \theta) &\stackrel{?}{=} -\cos \theta && \text{Original equation} \\ \cos 180° \cos \theta - \sin 180° \sin \theta &\stackrel{?}{=} -\cos \theta && \text{Sum of angles formula} \\ (-1) \cos \theta - 0 \sin \theta &\stackrel{?}{=} -\cos \theta && \text{Evaluate each expression.} \\ -\cos \theta &= -\cos \theta && \text{Simplify.}\end{aligned}$$

Check for Understanding

Concept Check

1. **Determine** whether $\sin (\alpha + \beta) = \sin \alpha + \sin \beta$ is an identity.
2. **Describe** a method for finding the exact value of sin 105°. Then find the value.
3. **OPEN ENDED** Determine whether $\cos (\alpha - \beta) < 1$ is *sometimes*, *always*, or *never* true. Explain your reasoning.

Guided Practice

Find the exact value of each expression.

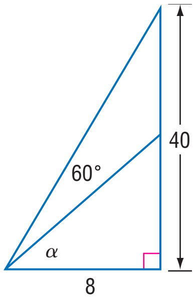
4. sin 75°
5. sin 165°
6. cos 255°
7. cos (−30°)
8. sin (−240°)
9. cos (−120°)

Verify that each of the following is an identity.

10. $\cos (270° - \theta) = -\sin \theta$
11. $\sin \left(\theta + \frac{\pi}{2}\right) = \cos \theta$
12. $\sin (\theta + 30°) + \cos (\theta + 60°) = \cos \theta$

CONTENTS

Application 13. **GEOMETRY** Determine the exact value of tan α.



Practice and Apply

Homework Help

For Exercises	See Examples
14–27	1
28–39	3
40, 41, 43–46	2

Extra Practice

See page 860.

Find the exact value of each expression.

14. sin 135°
15. cos 105°
16. sin 285°
17. cos 165°
18. cos 195°
19. sin 255°
20. cos 225°
21. sin 315°
22. sin (−15°)
23. cos (−45°)
24. cos (−150°)
25. sin (−165°)

26. What is the exact value of sin 75° − sin 15°?

27. Find the exact value of cos 105° + cos 225°.

Verify that each of the following is an identity.

28. $\sin(270° - \theta) = -\cos\theta$

29. $\cos(90° + \theta) = -\sin\theta$

30. $\cos(90° - \theta) = \sin\theta$

31. $\sin(90° - \theta) = \cos\theta$

32. $\sin\left(\theta + \frac{3\pi}{2}\right) = -\cos\theta$

33. $\cos(\pi - \theta) = -\cos\theta$

34. $\cos(2\pi + \theta) = \cos\theta$

35. $\sin(\pi - \theta) = \sin\theta$

36. $\sin(60° + \theta) + \sin(60° - \theta) = \sqrt{3}\cos\theta$

37. $\sin\left(\theta + \frac{\pi}{3}\right) - \cos\left(\theta + \frac{\pi}{6}\right) = \sin\theta$

38. $\sin(\alpha + \beta)\sin(\alpha - \beta) = \sin^2\alpha - \sin^2\beta$

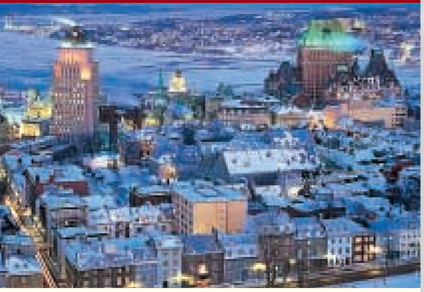
39. $\cos(\alpha + \beta) = \dfrac{1 - \tan\alpha\tan\beta}{\sec\alpha\sec\beta}$

COMMUNICATION **For Exercises 40 and 41, use the following information.**
A radio transmitter sends out two signals, one for voice communication and another for data. Suppose the equation of the voice wave is $v = 10\sin(2t - 30°)$ and the equation of the data wave is $d = 10\cos(2t + 60°)$.

40. Draw a graph of the waves when they are combined.

41. Refer to the application at the beginning of the lesson. What type of interference results? Explain.

More About. . .



Physics

In the northern hemisphere, the day with the least number of hours of daylight is December 21 or 22, the first day of winter.

Source: www.infoplease.com

PHYSICS **For Exercises 42–45, use the following information.**
On December 22, the maximum amount of light energy that falls on a square foot of ground at a certain location is given by $E\sin(113.5° + \phi)$, where ϕ is the latitude of the location. Use the sum of angles formula to find the amount of light energy, in terms of E, for each location.

42. Salem, OR (Latitude: 44.9° N)

43. Chicago, IL (Latitude: 41.8° N)

44. Charleston, SC (Latitude: 28.5°N)

45. San Diego, CA (Latitude 32.7° N)

46. **CRITICAL THINKING** Use the sum and difference formulas for sine and cosine to derive formulas for tan $(\alpha + \beta)$ and tan $(\alpha - \beta)$.

www.algebra2.com/self-check_quiz

47. **WRITING IN MATH** Answer the question that was posed at the beginning of the lesson.

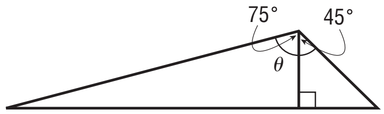
How are the sum and difference formulas used to describe communication interference?

Include the following in your answer:

- an explanation of the difference between constructive and destructive interference, and
- a description of how you would explain wave interference to a friend.

Standardized Test Practice

48. Find the exact value of sin θ.



(A) $\frac{\sqrt{3}}{2}$ (B) $\frac{\sqrt{2}}{2}$

(C) $\frac{1}{2}$ (D) $\frac{\sqrt{3}}{3}$

49. Find the exact value of cos $(-210°)$.

(A) $\frac{\sqrt{3}}{2}$ (B) 0.5 (C) $-\frac{\sqrt{3}}{2}$ (D) -0.5

Maintain Your Skills

Mixed Review

Verify that each of the following is an identity. *(Lesson 14-4)*

50. $\cot \theta + \sec \theta = \frac{\cos^2 \theta + \sin \theta}{\sin \theta \cos \theta}$

51. $\sin^2 \theta + \tan^2 \theta = (1 - \cos^2 \theta) + \frac{\sec^2 \theta}{\csc^2 \theta}$

52. $\sin \theta (\sin \theta + \csc \theta) = 2 - \cos^2 \theta$

53. $\frac{\sec \theta}{\tan \theta} = \csc \theta$

Simplify each expression. *(Lesson 14-3)*

54. $\frac{\tan \theta \csc \theta}{\sec \theta}$

55. $4\left(\sec^2 \theta - \frac{\sin^2 \theta}{\cos^2 \theta}\right)$

56. $(\cot \theta + \tan \theta)\sin \theta$

57. $\csc \theta \tan \theta + \sec \theta$

Find the exact values of the six trigonometric functions of θ if the terminal side of θ in standard position contains the given point. *(Lesson 13-3)*

58. $(5, -3)$

59. $(-3, -4)$

60. $(0, 2)$

Evaluate each expression. *(Lesson 12-2)*

61. $P(6, 4)$

62. $P(12, 7)$

63. $C(8, 3)$

64. $C(10, 4)$

65. **AVIATION** A pilot is flying from Chicago to Columbus, a distance of 300 miles. In order to avoid an area of thunderstorms, she alters her initial course by 15° and flies on this course for 75 miles. How far is she from Columbus? *(Lesson 13-5)*

66. Write $6y^2 - 34x^2 = 204$ in standard form. *(Lesson 8-5)*

Getting Ready for the Next Lesson

PREREQUISITE SKILL **Solve each equation.**

(To review ***solving equations using the Square Root Property****, see Lesson 6-4.)*

67. $x^2 = \frac{20}{16}$

68. $x^2 = \frac{9}{25}$

69. $x^2 = \frac{5}{25}$

70. $x^2 = \frac{18}{32}$

71. $x^2 - 1 = \frac{1}{2}$

72. $x^2 - 1 = \frac{4}{5}$

73. $x^2 = \frac{\sqrt{3}}{2} - \frac{1}{2}$

74. $x^2 = \frac{\sqrt{2}}{2} - 1$

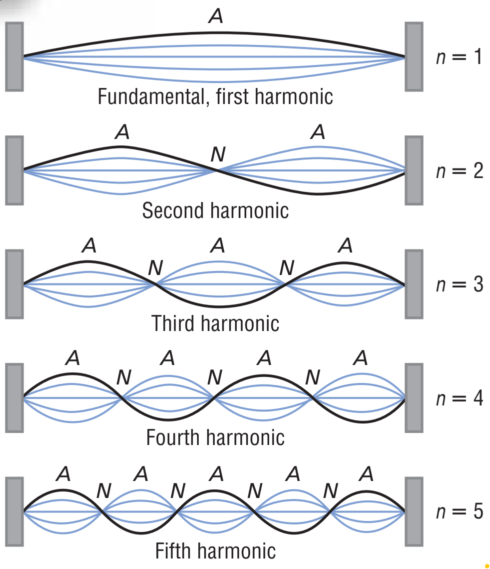
14-6 Double-Angle and Half-Angle Formulas

What You'll Learn

- Find values of sine and cosine involving double-angle formulas.
- Find values of sine and cosine involving half-angle formulas.

How can trigonometric functions be used to describe music?

Stringed instruments such as a piano, guitar, or violin rely on waves to produce the tones we hear. When the strings are struck or plucked, they vibrate. If the motion of the strings were observed in slow motion, you could see that there are places on the string, called *nodes*, that do not move under the vibration. Halfway between each pair of consecutive nodes are *antinodes* that undergo the maximum vibration. The nodes and antinodes form *harmonics*. These harmonics can be represented using variations of the equations $y = \sin 2\theta$ and $y = \sin \frac{1}{2}\theta$.



DOUBLE-ANGLE FORMULAS You can use the formula for $\sin(\alpha + \beta)$ to find the sine of twice an angle θ, $\sin 2\theta$, and the formula for $\cos(\alpha + \beta)$ to find the cosine of twice an angle θ, $\cos 2\theta$.

$$\begin{aligned}\sin 2\theta &= \sin(\theta + \theta)\\ &= \sin\theta\cos\theta + \cos\theta\sin\theta\\ &= 2\sin\theta\cos\theta\end{aligned}$$

$$\begin{aligned}\cos 2\theta &= \cos(\theta + \theta)\\ &= \cos\theta\cos\theta - \sin\theta\sin\theta\\ &= \cos^2\theta - \sin^2\theta\end{aligned}$$

You can find alternate forms for $\cos 2\theta$ by making substitutions into the expression $\cos^2\theta - \sin^2\theta$.

$$\begin{aligned}\cos^2\theta - \sin^2\theta &= (1 - \sin^2\theta) - \sin^2\theta && \text{Substitute } 1 - \sin^2\theta \text{ for } \cos^2\theta.\\ &= 1 - 2\sin^2\theta && \text{Simplify.}\\ \cos^2\theta - \sin^2\theta &= \cos^2\theta - (1 - \cos^2\theta) && \text{Substitute } 1 - \cos^2\theta \text{ for } \sin^2\theta.\\ &= 2\cos^2\theta - 1 && \text{Simplify.}\end{aligned}$$

These formulas are called the **double-angle formulas**.

Key Concept — *Double-Angle Formulas*

The following identities hold true for all values of θ.

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$\cos 2\theta = 1 - 2\sin^2\theta$$

$$\cos 2\theta = 2\cos^2\theta - 1$$

Example 1 Double-Angle Formulas

Find the exact value of each expression if $\sin\theta = \frac{4}{5}$ and θ is between 90° and 180°.

a. $\sin 2\theta$

Use the identity $\sin 2\theta = 2\sin\theta\cos\theta$.

First, find the value of $\cos\theta$.

$\cos^2\theta = 1 - \sin^2\theta$ — $\cos^2\theta + \sin^2\theta = 1$

$\cos^2\theta = 1 - \left(\frac{4}{5}\right)^2$ — $\sin\theta = \frac{4}{5}$

$\cos^2\theta = \frac{9}{25}$ — Subtract.

$\cos\theta = \pm\frac{3}{5}$ — Find the square root of each side.

Since θ is in the second quadrant, cosine is negative. Thus, $\cos\theta = -\frac{3}{5}$.

Now find $\sin 2\theta$.

$\sin 2\theta = 2\sin\theta\cos\theta$ — Double-angle formula

$\sin 2\theta = 2\left(\frac{4}{5}\right)\left(-\frac{3}{5}\right)$ — $\sin\theta = \frac{4}{5}$, $\cos\theta = -\frac{3}{5}$

$= -\frac{24}{25}$ — The value of $\sin 2\theta$ is $-\frac{24}{25}$.

b. $\cos 2\theta$

Use the identity $\cos 2\theta = 1 - 2\sin^2\theta$.

$\cos 2\theta = 1 - 2\sin^2\theta$ — Double-angle formula

$= 1 - 2\left(\frac{4}{5}\right)^2$ — $\sin\theta = \frac{4}{5}$

$= -\frac{7}{25}$ — The value of $\cos 2\theta$ is $-\frac{7}{25}$.

HALF-ANGLE FORMULAS

You can derive formulas for the sine and cosine of half a given angle using the double-angle formulas.

Find $\sin\frac{\alpha}{2}$.

$1 - 2\sin^2\theta = \cos 2\theta$ — Double-angle formula

$1 - 2\sin^2\frac{\alpha}{2} = \cos\alpha$ — Substitute $\frac{\alpha}{2}$ for θ and α for 2θ.

$\sin^2\frac{\alpha}{2} = \frac{1 - \cos\alpha}{2}$ — Solve for $\sin^2\frac{\alpha}{2}$.

$\sin\frac{\alpha}{2} = \pm\sqrt{\frac{1 - \cos\alpha}{2}}$ — Take the square root of each side.

Find $\cos\frac{\alpha}{2}$.

$2\cos^2\theta - 1 = \cos 2\theta$ — Double-angle formula

$2\cos^2\frac{\alpha}{2} - 1 = \cos\alpha$ — Substitute $\frac{\alpha}{2}$ for θ and α for 2θ.

$\cos^2\frac{\alpha}{2} = \frac{1 + \cos\alpha}{2}$ — Solve for $\cos^2\frac{\alpha}{2}$.

$\cos\frac{\alpha}{2} = \pm\sqrt{\frac{1 + \cos\alpha}{2}}$ — Take the square root of each side.

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These are called the **half-angle formulas**. The signs are determined by the function of $\frac{\alpha}{2}$.

Key Concept — *Half-Angle Formulas*

The following identities hold true for all values of α.

$$\sin \frac{\alpha}{2} = \pm\sqrt{\frac{1 - \cos \alpha}{2}} \qquad \cos \frac{\alpha}{2} = \pm\sqrt{\frac{1 + \cos \alpha}{2}}$$

Example 2 Half-Angle Formulas

Find $\cos \frac{\alpha}{2}$ if $\sin \alpha = -\frac{3}{4}$ and α is in the third quadrant.

Since $\cos \frac{\alpha}{2} = \pm\sqrt{\frac{1 + \cos \alpha}{2}}$, we must find $\cos \alpha$ first.

$\cos^2 \alpha = 1 - \sin^2 \alpha$ $\quad$ $\cos^2 \alpha + \sin^2 \alpha = 1$

$\cos^2 \alpha = 1 - \left(-\frac{3}{4}\right)^2$ $\quad$ $\sin \alpha = -\frac{3}{4}$

$\cos^2 \alpha = \frac{7}{16}$ $\quad$ Simplify.

$\cos \alpha = \pm\frac{\sqrt{7}}{4}$ $\quad$ Take the square root of each side.

Since α is in the third quadrant, $\cos \alpha = -\frac{\sqrt{7}}{4}$.

$\cos \frac{\alpha}{2} = \pm\sqrt{\frac{1 + \cos \alpha}{2}}$ $\quad$ Half-angle formula

$= \pm\sqrt{\frac{1 - \frac{\sqrt{7}}{4}}{2}}$ $\quad$ $\cos \alpha = -\frac{\sqrt{7}}{4}$

$= \pm\sqrt{\frac{4 - \sqrt{7}}{8}}$ $\quad$ Simplify the radicand.

$= \pm\frac{\sqrt{4 - \sqrt{7}}}{2\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}}$ $\quad$ Rationalize.

$= \pm\frac{\sqrt{8 - 2\sqrt{7}}}{4}$ $\quad$ Multiply.

Since α is between 180° and 270°, $\frac{\alpha}{2}$ is between 90° and 135°. Thus, $\cos \frac{\alpha}{2}$ is negative and equals $-\frac{\sqrt{8 - 2\sqrt{7}}}{4}$.

Study Tip

Choosing the Sign

You may want to determine the quadrant in which the terminal side of $\frac{\alpha}{2}$ will lie in the first step of the solution. Then you can use the correct sign from the beginning.

Example 3 Evaluate Using Half-Angle Formulas

Find the exact value of each expression by using the half-angle formulas.

a. sin 105°

$\sin 105° = \sin \frac{210°}{2}$

$= \sqrt{\frac{1 - \cos 210°}{2}}$ $\quad$ $\sin \frac{\alpha}{2} = \pm\sqrt{\frac{1 - \cos \alpha}{2}}$

(continued on the next page)

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$= \sqrt{\dfrac{1-\left(-\frac{\sqrt{3}}{2}\right)}{2}}$ cos 210° $= -\frac{\sqrt{3}}{2}$

$= \sqrt{\dfrac{2+\sqrt{3}}{4}}$ Simplify the radicand.

$= \dfrac{\sqrt{2+\sqrt{3}}}{2}$ Simplify the denominator.

b. $\cos \frac{\pi}{8}$

$\cos \frac{\pi}{8} = \dfrac{\frac{\pi}{4}}{2}$

$= \sqrt{\dfrac{1+\cos\frac{\pi}{4}}{2}}$ $\cos \frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\alpha}{2}}$

$= \sqrt{\dfrac{1+\frac{\sqrt{2}}{2}}{2}}$ $\cos \frac{\pi}{4} = \frac{\sqrt{2}}{2}$

$= \sqrt{\dfrac{2+\sqrt{2}}{4}}$ Simplify the radicand.

$= \dfrac{\sqrt{2+\sqrt{2}}}{2}$ Simplify the denominator.

Recall that you can use the sum and difference formulas to verify identities. Double- and half-angle formulas can also be used to verify identities.

Example 4 Verify Identities

Verify that $(\sin\theta + \cos\theta)^2 = 1 + \sin 2\theta$ is an identity.

$(\sin\theta + \cos\theta)^2 \stackrel{?}{=} 1 + \sin 2\theta$ Original equation

$\sin^2\theta + 2\sin\theta\cos\theta + \cos^2\theta \stackrel{?}{=} 1 + \sin 2\theta$ Multiply.

$1 + 2\sin\theta\cos\theta \stackrel{?}{=} 1 + \sin 2\theta$ $\sin^2\theta + \cos^2\theta = 1$

$1 + \sin 2\theta = 1 + \sin 2\theta$ Double-angle formula

Check for Understanding

Concept Check

1. **Explain** how to find $\cos \frac{x}{2}$ if x is in the third quadrant.

2. **Find a counterexample** to show that $\cos 2\theta = 2\cos\theta$ is not an identity.

3. **OPEN ENDED** Describe the conditions under which you would use each of the three identities for $\cos 2\theta$.

Guided Practice

Find the exact values of $\sin 2\theta$, $\cos 2\theta$, $\sin \frac{\theta}{2}$, and $\cos \frac{\theta}{2}$ for each of the following.

4. $\cos\theta = \frac{3}{5}$; $0° < \theta < 90°$

5. $\cos\theta = -\frac{2}{3}$; $180° < \theta < 270°$

6. $\sin\theta = \frac{1}{2}$; $0° < \theta < 90°$

7. $\sin\theta = -\frac{3}{4}$; $270° < \theta < 360°$

Find the exact value of each expression by using the half-angle formulas.

8. sin 195°

9. $\cos \frac{19\pi}{12}$

CONTENTS

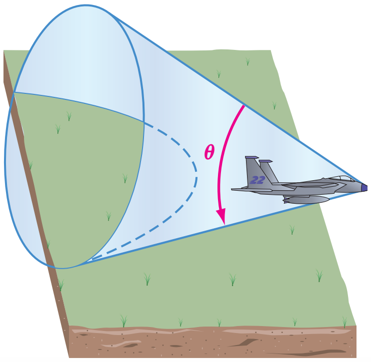
Verify that each of the following is an identity.

10. $\cot x = \dfrac{\sin 2x}{1 - \cos 2x}$

11. $\cos^2 2x + 4 \sin^2 x \cos^2 x = 1$

Application

12. AVIATION When a jet travels at speeds greater than the speed of sound, a sonic boom is created by the sound waves forming a cone behind the jet. If θ is the measure of the angle at the vertex of the cone, then the Mach number M can be determined using the formula $\sin \frac{\theta}{2} = \frac{1}{M}$. Find the Mach number of a jet if a sonic boom is created by a cone with a vertex angle of 75°.



Practice and Apply

Homework Help

For Exercises	See Examples
13–24, 38, 39	1, 2
25–30, 37	3
31–36	4

Extra Practice

See page 861.

Find the exact values of sin 2θ, cos 2θ, sin $\frac{\theta}{2}$, and cos $\frac{\theta}{2}$ for each of the following.

13. $\sin \theta = \frac{5}{13}$; $90° < \theta < 180°$

14. $\cos \theta = \frac{1}{5}$; $270° < \theta < 360°$

15. $\cos \theta = -\frac{1}{3}$; $180° < \theta < 270°$

16. $\sin \theta = -\frac{3}{5}$; $180° < \theta < 270°$

17. $\sin \theta = -\frac{3}{8}$; $270° < \theta < 360°$

18. $\cos \theta = -\frac{1}{4}$; $90° < \theta < 180°$

19. $\cos \theta = \frac{1}{6}$; $0° < \theta < 90°$

20. $\cos \theta = -\frac{12}{13}$; $180° < \theta < 270°$

21. $\sin \theta = -\frac{1}{3}$; $270° < \theta < 360°$

22. $\sin \theta = -\frac{1}{4}$; $180° < \theta < 270°$

23. $\cos \theta = \frac{2}{3}$; $0° < \theta < 90°$

24. $\sin \theta = \frac{2}{5}$; $90° < \theta < 180°$

Find the exact value of each expression by using the half-angle formulas.

25. $\cos 165°$

26. $\sin 22\frac{1}{2}°$

27. $\cos 157\frac{1}{2}°$

28. $\sin 345°$

29. $\sin \frac{7\pi}{8}$

30. $\cos \frac{7\pi}{12}$

Verify that each of the following is an identity.

31. $\sin 2x = 2 \cot x \sin^2 x$

32. $2 \cos^2 \frac{x}{2} = 1 + \cos x$

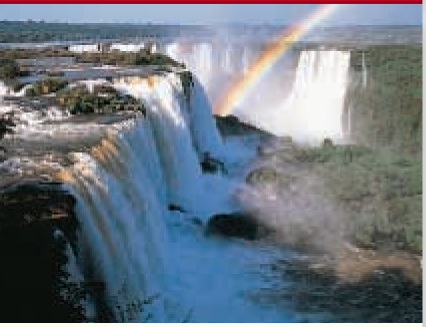
33. $\sin^4 x - \cos^4 x = 2 \sin^2 x - 1$

34. $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$

35. $\tan^2 \frac{x}{2} = \dfrac{1 - \cos x}{1 + \cos x}$

36. $\dfrac{1}{\sin x \cos x} - \dfrac{\cos x}{\sin x} = \tan x$

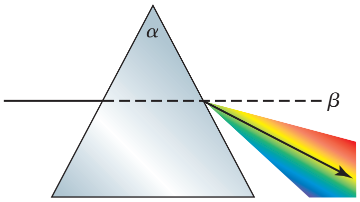
More About. . .



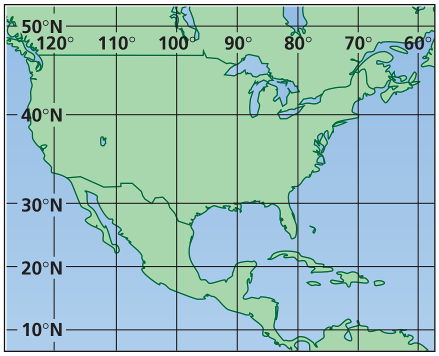
Optics

A rainbow appears when the sun shines through water droplets that act as a prism.

37. OPTICS If a glass prism has an apex angle of measure α and an angle of deviation of measure β, then the index of refraction n of the prism is given by $n = \dfrac{\sin\left[\frac{1}{2}(\alpha + \beta)\right]}{\sin \frac{\alpha}{2}}$.



What is the angle of deviation of a prism with an apex angle of 40° and an index of refraction of 2?



GEOGRAPHY **For Exercises 38 and 39, use the following information.** A Mercator projection map uses a flat projection of Earth in which the distance between the lines of latitude increases with their distance from the equator. The calculation of the location of a point on this projection uses the expression $\tan\left(45° + \frac{L}{2}\right)$, where L is the latitude of the point.

38. Write this expression in terms of a trigonometric function of L.

39. Find the exact value of the expression if $L = 60°$.

PHYSICS **For Exercises 40 and 41, use the following information.** An object is propelled from ground level with an initial velocity of v at an angle of elevation θ.

40. The horizontal distance d it will travel can be determined using $d = \frac{v^2 \sin 2\theta}{g}$, where g is acceleration due to gravity. Verify that this expression is the same as $\frac{2}{g}v^2(\tan\theta - \tan\theta \sin^2\theta)$.

41. The maximum height h the object will reach can be determined using the formula $h = \frac{v^2 \sin^2\theta}{2g}$. Find the ratio of the maximum height attained to the horizontal distance traveled.

CRITICAL THINKING **For Exercises 42–46, use the following information.** Consider the functions $f(x) = \sin 2x$, $g(x) = \sin^2 x$, $h(x) = -\cos^2 x$, and $k(x) = -\frac{1}{2}\cos 2x$.

42. Draw the graphs of $y = g(x)$, $y = h(x)$, and $y = k(x)$ on the same coordinate plane on the interval from $x = -2\pi$ to $x = 2\pi$. What do you notice about the graphs?

43. Where do the maxima and minima of g, h, and k occur?

44. Draw the graph of $y = f(x)$ on a separate coordinate plane.

45. What is the behavior of the graph of $f(x)$ at the locations found in Exercise 43?

46. Use what you know about transformations to determine c and d so that $g(x) = h(x) + c = k(x) + d$.

47. **WRITING IN MATH** Answer the question that was posed at the beginning of the lesson.

How can trigonometric functions be used to describe music?

Include the following in your answer:

- a description of what happens to the graph of the function of a vibrating string as it moves from one harmonic to the next, and
- an explanation of what happens to the period of the function as you move from the nth harmonic to the $(n + 1)$th harmonic.

Standardized Test Practice

48. Find the exact value of $\cos 2\theta$ if $\sin\theta = \frac{-\sqrt{5}}{3}$ and $180° < \theta < 270°$.

Ⓐ $\frac{-\sqrt{6}}{6}$ Ⓑ $\frac{-\sqrt{30}}{6}$ Ⓒ $\frac{-4\sqrt{5}}{9}$ Ⓓ $\frac{-1}{9}$

49. Find the exact value of $\sin\frac{\theta}{2}$ if $\cos\theta = \frac{\sqrt{3}}{2}$ and $0° < \theta < 90°$.

Ⓐ $\frac{\sqrt{3}}{2}$ Ⓑ $\frac{\sqrt{2-\sqrt{3}}}{2}$ Ⓒ $\frac{\sqrt{2+\sqrt{3}}}{2}$ Ⓓ $\frac{1}{2}$

CONTENTS

Maintain Your Skills

Mixed Review

Find the exact value of each expression. *(Lesson 14-5)*

50. $\cos 15°$

51. $\sin 15°$

52. $\sin (-135°)$

53. $\cos 150°$

54. $\sin 105°$

55. $\cos (-300°)$

Verify that each of the following is an identity. *(Lesson 14-4)*

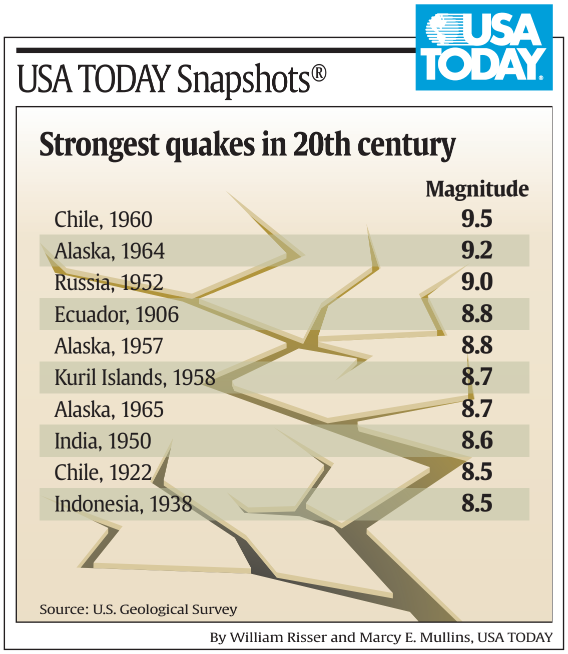
56. $\cot^2 \theta - \sin^2 \theta = \dfrac{\cos^2 \theta \csc^2 \theta - \sin^2 \theta}{\sin^2 \theta \csc^2 \theta}$

57. $\cos \theta (\cos \theta + \cot \theta) = \cot \theta \cos \theta (\sin \theta + 1)$

EARTHQUAKE **For Exercises 58 and 59, use the following information.** The magnitude of an earthquake M measured on the Richter scale is given by $M = \log_{10} x$, where x represents the amplitude of the seismic wave causing ground motion. *(Lesson 10-2)*

58. How many times as great was the 1960 Chile earthquake as the 1938 Indonesia earthquake?

59. The largest aftershock of the 1964 Alaskan earthquake was 6.7 on the Richter scale. How many times as great was the main earthquake as this aftershock?



USA TODAY Snapshots®

Strongest quakes in 20th century

	Magnitude
Chile, 1960	9.5
Alaska, 1964	9.2
Russia, 1952	9.0
Ecuador, 1906	8.8
Alaska, 1957	8.8
Kuril Islands, 1958	8.7
Alaska, 1965	8.7
India, 1950	8.6
Chile, 1922	8.5
Indonesia, 1938	8.5

Source: U.S. Geological Survey

By William Risser and Marcy E. Mullins, USA TODAY

Getting Ready for the Next Lesson

PREREQUISITE SKILL **Solve each equation.**

*(To review **solving equations using the Zero Product Property**, see Lesson 6-3.)*

60. $(x + 6)(x - 5) = 0$

61. $(x - 1)(x + 1) = 0$

62. $x(x + 2) = 0$

63. $(2x - 5)(x + 2) = 0$

64. $(2x + 1)(2x - 1) = 0$

65. $x^2(2x + 1) = 0$

Practice Quiz 2

Lessons 14-4 through 14-6

Verify that each of the following is an identity. *(Lessons 14-5)*

1. $\sin \theta \sec \theta = \tan \theta$

2. $\sec \theta - \cos \theta = \sin \theta \tan \theta$

3. $\sin \theta + \tan \theta = \dfrac{\sin \theta (\cos \theta + 1)}{\cos \theta}$

Verify that each of the following is an identity. *(Lessons 14-4 and 14-5)*

4. $\sin (90° + \theta) = \cos \theta$

5. $\cos \left(\dfrac{3\pi}{2} - \theta\right) = -\sin \theta$

6. $\sin (\theta + 30°) + \cos (\theta + 60°) = \cos \theta$

Find the exact value of each expression by using the double-angle or half-angle formulas. *(Lesson 14-6)*

7. $\sin 2\theta$ if $\cos \theta = -\dfrac{\sqrt{3}}{2}$; $180° < \theta < 270°$

8. $\cos \dfrac{\theta}{2}$ if $\sin \theta = -\dfrac{9}{41}$; $270° \leq \theta < 360°$

9. $\sin 165°$

10. $\cos \dfrac{5\pi}{8}$

Graphing Calculator Investigation

A Preview of Lesson 14-7

Solving Trigonometric Equations

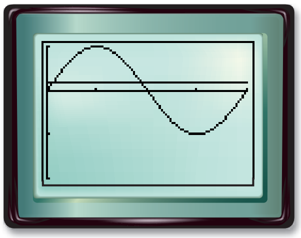
The graph of a trigonometric function is made up of points that represent all values that satisfy the function. To solve a trigonometric equation, you need to find all values of the variable that satisfy the equation. You can use a TI-83 Plus to solve trigonometric equations by graphing each side of the equation as a function and then locating the points of intersection.

Example 1 **Use a graphing calculator to solve $\sin x = 0.2$ if $0° \le x < 360°$.**

Rewrite the equation as two functions, $y = \sin x$ and $y = 0.2$. Then graph the two functions. Look for the point of intersection.

Make sure that your calculator is in degree mode to get the correct viewing window.

KEYSTROKES: MODE ▼ ▼ ▶ ENTER WINDOW 0 ENTER 360 ENTER 90 ENTER −2 ENTER 1 ENTER 1 ENTER Y= SIN X,T,θ,n ENTER 0.2 ENTER GRAPH



[0, 360] scl: 90 by [−2, 1] scl: 1

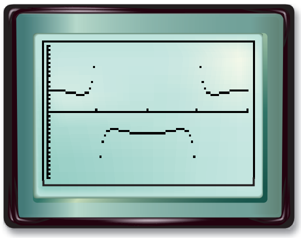
Based on the graph, you can see that there are two points of intersection in the interval 0°≤ x < 360°. Use Zoom or 2nd [CALC] 5 to approximate the solutions. The approximate solutions are 168.5° and 11.5°.

Like other equations you have studied, some trigonometric equations have no real solutions. Carefully examine the graphs over their respective periods for points of intersection. If there are no points of intersection, then the trigonometric equation has no real solutions.

Example 2 **Use a graphing calculator to solve $\tan^2 x \cos x + 5 \cos x = 0$ if $0° \le x < 360°$.**

Because the tangent function is not continuous, place the calculator in **Dot** mode. The related functions to be graphed are $y = \tan^2 x \cos x + 5 \cos x$ and $y = 0$.

These two functions do not intersect. Therefore, the equation $\tan^2 x \cos x + 5 \cos x = 0$ has no real solutions.



[0, 360] scl: 90 by [−15, 15] scl: 1

Exercises

Use a graphing calculator to solve each equation for the values of x indicated.

1. $\sin x = 0.8$ if $0° \le x < 360°$

2. $\tan x = \sin x$ if $0° \le x < 360°$

3. $2 \cos x + 3 = 0$ if $0° \le x < 360°$

4. $0.5 \cos x = 1.4$ if $-720° \le x < 720°$

5. $\sin 2x = \sin x$ if $0° \le x < 360°$

6. $\sin 2x - 3 \sin x = 0$ if $-360° \le x < 360°$

www.algebra2.com/other_calculator_keystrokes

CONTENTS

14-7 Solving Trigonometric Equations

What You'll Learn

- Solve trigonometric equations.
- Use trigonometric equations to solve real-world problems.

Vocabulary

- trigonometric equation

How can trigonometric equations be used to predict temperature?

The average daily high temperature for a region can be described by a trigonometric function. For example, the average daily high temperature for each month in Orlando, Florida, can be modeled by the function $T = 11.56 \sin (0.4516x - 1.641) + 80.89$, where T represents the average daily high temperature in degrees Fahrenheit and x represents the month of the year. This equation can be used to predict the months in which the average temperature in Orlando will be at or above a desired temperature.

SOLVE TRIGONOMETRIC EQUATIONS You have seen that trigonometric identities are true for *all* values of the variable for which the equation is defined. However, most **trigonometric equations** like some algebraic equations, are true for *some* but not *all* values of the variable.

Example 1 Solve Equations for a Given Interval

Find all solutions of each equation for the given interval.

a. $\cos^2 \theta = 1$; $0° \le \theta < 360°$

$$\cos^2 \theta = 1 \quad \text{Original equation}$$
$$\cos^2 \theta - 1 = 0 \quad \text{Solve for 0.}$$
$$(\cos \theta + 1)(\cos \theta - 1) = 0 \quad \text{Factor.}$$

Now use the Zero Product Property.

$$\cos \theta + 1 = 0 \quad \text{or} \quad \cos \theta - 1 = 0$$
$$\cos \theta = -1 \qquad\qquad \cos \theta = 1$$
$$\theta = 180° \qquad\qquad \theta = 0°$$

The solutions are 0° and 180°.

b. $\sin 2\theta = 2 \cos \theta$; $0 \le \theta < 2\pi$

$$\sin 2\theta = 2 \cos \theta \quad \text{Original equation}$$
$$2 \sin \theta \cos \theta = 2 \cos \theta \quad \sin 2\theta = 2 \sin \theta \cos \theta$$
$$2 \sin \theta \cos \theta - 2 \cos \theta = 0 \quad \text{Solve for 0.}$$
$$2 \cos \theta\,(\sin \theta - 1) = 0 \quad \text{Factor.}$$

(continued on the next page)

Use the Zero Product Property.

$2\cos\theta = 0$ or $\sin\theta - 1 = 0$

$\cos\theta = 0$ $\quad\sin\theta = 1$

$\theta = \frac{\pi}{2}$ or $\frac{3\pi}{2}$ $\quad\theta = \frac{\pi}{2}$

The solutions are $\frac{\pi}{2}$ and $\frac{3\pi}{2}$.

Trigonometric equations are usually solved for values of the variable between 0° and 360° or 0 radians and 2π radians. There are solutions outside that interval. These other solutions differ by integral multiples of the period of the function.

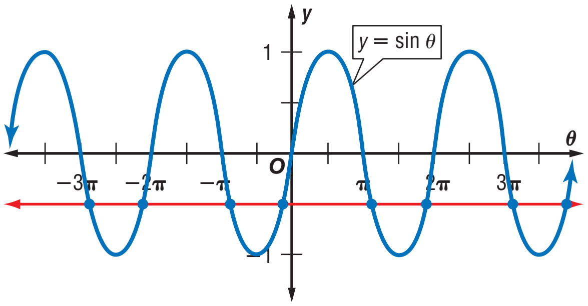
Example 2 Solve Trigonometric Equations

a. Solve $2\sin\theta = -1$ for all values of θ if θ is measured in radians.

$2\sin\theta = -1$ Original equation

$\sin\theta = -\frac{1}{2}$ Divide each side by 2.

Look at the graph of $y = \sin\theta$ to find solutions of $\sin\theta = -\frac{1}{2}$.



The solutions are $\frac{7\pi}{6}, \frac{11\pi}{6}, \frac{19\pi}{6}, \frac{23\pi}{6}$, and so on, and $-\frac{7\pi}{6}, -\frac{11\pi}{6}, -\frac{19\pi}{6}, -\frac{23\pi}{6}$, and so on. The only solutions in the interval 0 to 2π are $\frac{7\pi}{6}$ and $\frac{11\pi}{6}$. The period of the sine function is 2π radians. So the solutions can be written as $\frac{7\pi}{6} + 2k\pi$ and $\frac{11\pi}{6} + 2k\pi$, where k is any integer.

b. Solve $\cos 2\theta + \cos\theta + 1 = 0$ for all values of θ if θ is measured in degrees.

$\cos 2\theta + \cos\theta + 1 = 0$ Original equation

$2\cos^2\theta - 1 + \cos\theta + 1 = 0$ $\cos 2\theta = 2\cos^2\theta - 1$

$2\cos^2\theta + \cos\theta = 0$ Simplify.

$\cos\theta\,(2\cos\theta + 1) = 0$ Factor.

Solve for θ in the interval 0° to 360°.

$\cos\theta = 0$ or $2\cos\theta + 1 = 0$

$\theta = 90°$ or $270°$ $\quad 2\cos\theta = -1$

$\cos\theta = -\frac{1}{2}$

$\theta = 120°$ or $240°$

The solutions are $90° + k \cdot 180°$, $120° + k \cdot 360°$, and $240° + k \cdot 360°$.

Study Tip

Expressing Solutions as Multiples

The expression $90° + k \cdot 180°$ includes 270° and its multiples, so it is not necessary to list them separately.

If an equation cannot be solved easily by factoring, try rewriting the expression using trigonometric identities. However, using identities and some algebraic operations, such as squaring, may result in extraneous solutions. So, it is necessary to check your solutions using the original equation.

CONTENTS

Example 3 Solve Trigonometric Equations Using Identities

Solve $\cos\theta \tan\theta - \sin^2\theta = 0$.

$$\begin{aligned}
\cos\theta \tan\theta - \sin^2\theta &= 0 && \text{Original equation}\\
\cos\theta\left(\frac{\sin\theta}{\cos\theta}\right) - \sin^2\theta &= 0 && \tan\theta = \frac{\sin\theta}{\cos\theta}\\
\sin\theta - \sin^2\theta &= 0 && \text{Multiply.}\\
\sin\theta\,(1 - \sin\theta) &= 0 && \text{Factor.}
\end{aligned}$$

$$\begin{aligned}
\sin\theta &= 0 &\quad \text{or} \quad 1 - \sin\theta &= 0\\
\theta &= 0°, 180°, \text{ or } 360° & \sin\theta &= 1\\
& & \theta &= 90°
\end{aligned}$$

CHECK

$$\begin{aligned}
\cos\theta \tan\theta - \sin^2\theta &= 0\\
\cos 0° \tan 0° - \sin^2 0° &\overset{?}{=} 0 && \theta = 0°\\
1 \cdot 0 - 0 &\overset{?}{=} 0\\
0 &= 0 \;\checkmark
\end{aligned}$$

$$\begin{aligned}
\cos\theta \tan\theta - \sin^2\theta &= 0\\
\cos 180° \tan 180° - \sin^2 180° &\overset{?}{=} 0 && \theta = 180°\\
-1 \cdot 0 - 0 &\overset{?}{=} 0\\
0 &= 0 \;\checkmark
\end{aligned}$$

$$\begin{aligned}
\cos\theta \tan\theta - \sin^2\theta &= 0\\
\cos 360° \tan 360° - \sin^2 360° &\overset{?}{=} 0 && \theta = 360°\\
1 \cdot 0 - 0 &\overset{?}{=} 0\\
0 &= 0 \;\checkmark
\end{aligned}$$

$$\begin{aligned}
\cos\theta \tan\theta - \sin^2\theta &= 0\\
\cos 90° \tan 90° - \sin^2 90° &\overset{?}{=} 0 && \theta = 90°
\end{aligned}$$

tan 90° is undefined.

Thus, 90° is not a solution.

The solution is $0° + k \cdot 180°$.

Some trigonometric equations have no solution. For example, the equation $\cos x = 4$ has no solution since all values of $\cos x$ are between -1 and 1, inclusive. Thus, the solution set for $\cos x = 4$ is empty.

Example 4 Determine Whether a Solution Exists

Solve $3\cos 2\theta - 5\cos\theta = 1$.

$$\begin{aligned}
3\cos 2\theta - 5\cos\theta &= 1 && \text{Original equation}\\
3(2\cos^2\theta - 1) - 5\cos\theta &= 1 && \cos 2\theta = 2\cos^2\theta - 1\\
6\cos^2\theta - 3 - 5\cos\theta &= 1 && \text{Multiply.}\\
6\cos^2\theta - 5\cos\theta - 4 &= 0 && \text{Subtract 1 from each side.}\\
(3\cos\theta - 4)(2\cos\theta + 1) &= 0 && \text{Factor.}
\end{aligned}$$

$$\begin{aligned}
3\cos\theta - 4 &= 0 &\quad \text{or} \quad 2\cos\theta + 1 &= 0\\
3\cos\theta &= 4 & 2\cos\theta &= -1\\
\cos\theta &= \frac{4}{3} & \cos\theta &= -\frac{1}{2}
\end{aligned}$$

Not possible since $\cos\theta$ cannot be greater than 1. $\theta = 120°$ or $240°$

Thus, the solutions are $120° + k \cdot 360°$ and $240° + k \cdot 360°$.

USE TRIGONOMETRIC EQUATIONS Trigonometric equations are often used to solve real-world situations.

Example 5 Use a Trigonometric Equation

GARDENING Rhonda wants to wait to plant her flowers until there are at least 14 hours of daylight. The number of hours of daylight H in her town can be represented by $H = 11.45 + 6.5 \sin (0.0168d - 1.333)$, where d is the day of the year and angle measures are in radians. On what day is it safe for Rhonda to plant her flowers?

$$\begin{aligned}
H &= 11.45 + 6.5 \sin (0.0168d - 1.333) && \text{Original equation} \\
14 &= 11.45 + 6.5 \sin (0.0168d - 1.333) && H = 14 \\
2.55 &= 6.5 \sin (0.0168d - 1.333) && \text{Subtract 11.45 from each side.} \\
0.392 &= \sin (0.0168d - 1.333) && \text{Divide each side by 6.5.} \\
0.403 &= 0.0168d - 1.333 && \text{Sin}^{-1}\ 0.392 = 0.403 \\
1.736 &= 0.0168d && \text{Add 1.333 to each side.} \\
103.333 &= d && \text{Divide each side by 0.0168.}
\end{aligned}$$

Rhonda can safely plant her flowers around the 104th day of the year, or around April 14.

Check for Understanding

Concept Check

1. **Tell** why the equation $\sec \theta = 0$ has no solutions.
2. **Explain** why the number of solutions to the equation $\sin \theta = \frac{\sqrt{3}}{2}$ is infinite.
3. **OPEN ENDED** Write an example of a trigonometric equation that has no solution.

Guided Practice

Find all solutions of each equation for the given interval.

4. $4 \cos^2 \theta = 1; 0° \leq \theta < 360°$
5. $2 \sin^2 \theta - 1 = 0; 90° < \theta < 270°$
6. $\sin 2\theta = \cos \theta; 0 \leq \theta < 2\pi$
7. $3 \sin^2 \theta - \cos^2 \theta = 0; 0 \leq \theta < \frac{\pi}{2}$

Solve each equation for all values of θ if θ is measured in radians.

8. $\cos 2\theta = \cos \theta$
9. $\sin \theta + \sin \theta \cos \theta = 0$

Solve each equation for all values of θ if θ is measured in degrees.

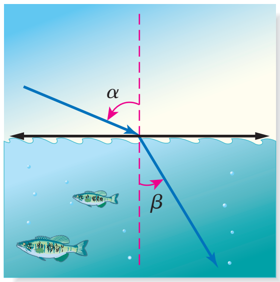
10. $\sin \theta = 1 + \cos \theta$
11. $2 \cos^2 \theta + 2 = 5 \cos \theta$

Solve each equation for all values of θ.

12. $2 \sin^2 \theta - 3 \sin \theta - 2 = 0$
13. $2 \cos^2 \theta + 3 \sin \theta - 3 = 0$

Application

14. **PHYSICS** According to Snell's law, the angle at which light enters water α is related to the angle at which light travels in water β by the equation $\sin \alpha = 1.33 \sin \beta$. At what angle does a beam of light enter the water if the beam travels at an angle of 23° through the water?



CONTENTS

Practice and Apply

Homework Help

For Exercises	See Examples
15–22	1
23–34	2
35–40	3, 4
41–43	5

Extra Practice

See page 861.

Find all solutions of each equation for the given interval.

15. $2 \cos \theta - 1 = 0$; $0° \leq \theta < 360°$

16. $2 \sin \theta = -\sqrt{3}$; $180° < \theta < 360°$

17. $4 \sin^2 \theta = 1$; $180° < \theta < 360°$

18. $4 \cos^2 \theta = 3$; $0° \leq \theta < 360°$

19. $2 \cos^2 \theta = \sin \theta + 1$; $0 \leq \theta < 2\pi$

20. $\sin^2 \theta - 1 = \cos^2 \theta$; $0 \leq \theta < \pi$

21. $2 \sin^2 \theta + \sin \theta = 0$; $\pi < \theta < 2\pi$

22. $2 \cos^2 \theta = -\cos \theta$; $0 \leq \theta < 2\pi$

Solve each equation for all values of θ if θ is measured in radians.

23. $\cos 2\theta + 3 \cos \theta - 1 = 0$

24. $2 \sin^2 \theta - \cos \theta - 1 = 0$

25. $\cos^2 \theta - \frac{5}{2} \cos \theta - \frac{3}{2} = 0$

26. $\cos \theta = 3 \cos \theta - 2$

27. $4 \cos^2 \theta - 4 \cos \theta + 1 = 0$

28. $\cos 2\theta = 1 - \sin \theta$

Solve each equation for all values of θ if θ is measured in degrees.

29. $\sin \theta = \cos \theta$

30. $\tan \theta = \sin \theta$

31. $\sin^2 \theta - 2 \sin \theta - 3 = 0$

32. $4 \sin^2 \theta - 4 \sin \theta + 1 = 0$

33. $\tan^2 \theta - \sqrt{3} \tan \theta = 0$

34. $\cos^2 \theta - \frac{7}{2} \cos \theta - 2 = 0$

Solve each equation for all values of θ.

35. $\sin^2 \theta + \cos 2\theta - \cos \theta = 0$

36. $2 \sin^2 \theta - 3 \sin \theta - 2 = 0$

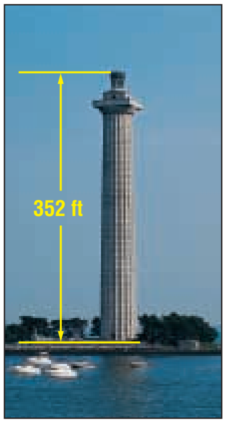
37. $\sin^2 \theta = \cos^2 \theta - 1$

38. $2 \cos^2 \theta + \cos \theta = 0$

39. $\sin \frac{\theta}{2} + \cos \theta = 1$

40. $\sin \frac{\theta}{2} + \cos \frac{\theta}{2} = \sqrt{2}$

LIGHT **For Exercises 41 and 42, use the information shown.**



41. The length of the shadow S of the International Peace Memorial at Put-In-Bay, Ohio, depends upon the angle of inclination of the Sun, θ. Express S as a function of θ.

42. Find the angle of inclination θ that will produce a shadow 560 feet long.

More About. . .

Waves

In the oceans, the height and period of water waves are determined by wind velocity, the duration of the wind, and the distance the wind has blown across the water.

Source: www.infoplease.com

WAVES **For Exercises 43 and 44, use the following information.**
For a short time after a wave is created by a boat, the height of the wave can be modeled using $y = \frac{1}{2}h + \frac{1}{2}h \sin \frac{2\pi t}{P}$, where h is the maximum height of the wave in feet, P is the period in seconds, and t is the propagation of the wave in seconds.

43. If $h = 3$ and $P = 2$ seconds, write the equation for the wave and draw its graph over a 10-second interval.

44. How many times over the first 10 seconds does the graph predict the wave to be one foot high?

45. CRITICAL THINKING Computer games often use transformations to distort images on the screen. In one such transformation, an image is rotated counterclockwise using the equations $x' = x \cos \theta - y \sin \theta$ and $y' = x \sin \theta + y \cos \theta$. If the coordinates of an image point are (3, 4) after a 60° rotation, what are the coordinates of the preimage point?

46. WRITING IN MATH Answer the question that was posed at the beginning of the lesson.

How can trigonometric equations be used to predict temperature?

Include the following in your answer:

- an explanation of why the sine function can be used to model the average daily temperature, and
- an explanation of why, during one period, you might find a specific average temperature twice.

Standardized Test Practice

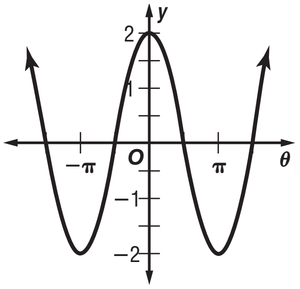
47. Which of the following is *not* a possible solution of $0 = \sin \theta + \cos \theta \tan^2 \theta$?

(A) $\frac{3\pi}{4}$ (B) $\frac{7\pi}{4}$ (C) 2π (D) $\frac{5\pi}{2}$

48. The graph of the equation $y = 2 \cos \theta$ is shown. Which is a solution for $2 \cos \theta = 1$?

(A) $\frac{8\pi}{3}$ (B) $\frac{13\pi}{3}$

(C) $\frac{10\pi}{3}$ (D) $\frac{15\pi}{3}$



Maintain Your Skills

Mixed Review

Find the exact value of $\sin 2\theta$, $\cos 2\theta$, $\sin \frac{\theta}{2}$, and $\cos \frac{\theta}{2}$ for each of the following. *(Lesson 14-6)*

49. $\sin \theta = \frac{3}{5}$; $0° < \theta < 90°$

50. $\cos \theta = \frac{1}{2}$; $0° < \theta < 90°$

51. $\cos \theta = \frac{5}{6}$; $0° < \theta < 90°$

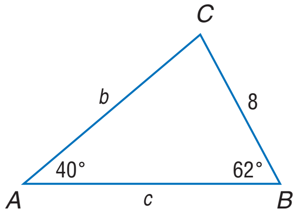
52. $\sin \theta = \frac{4}{5}$; $0° < \theta < 90°$

Find the exact value of each expression. *(Lesson 14-5)*

53. sin 240°

54. cos 315°

55. Solve $\triangle ABC$. Round measures of sides and angles to the nearest tenth. *(Lesson 13-4)*



WebQuest Internet Project

Trig Class Angles for Lessons in Lit

It is time to complete your project. Use the information and data you have gathered about the applications of trigonometry to prepare a poster, report, or Web page. Be sure to include graphs, tables, or diagrams in the presentation.

www.algebra2.com/webquest

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Chapter 14 Study Guide and Review

Vocabulary and Concept Check

amplitude (p. 763)
double-angle formula (p. 791)
half-angle formula (p. 793)
midline (p. 771)
phase shift (p. 769)
trigonometric equation (p. 799)
trigonometric identity (p. 777)
vertical shift (p. 771)

Choose the correct letter that best matches each phrase.

1. horizontal translation of a trigonometric function
2. a reference line about which a graph oscillates
3. vertical translation of a trigonometric function
4. the formula used to find cos $22\frac{1}{2}°$
5. $\sin 2\theta = 2 \sin \theta \cos \theta$
6. a measure of how long it takes for a graph to repeat itself
7. $\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$
8. the absolute value of half the difference between the maximum and minimum values of a periodic function

a. amplitude
b. midline
c. period
d. vertical shift
e. double-angle formula
f. half-angle formula
g. difference of angles formula
h. phase shift

Lesson-by-Lesson Review

14-1 Graphing Trigonometric Functions

See pages 762–768.

Concept Summary

- For trigonometric functions of the form $y = a \sin b\theta$ and $y = a \cos b\theta$, the amplitude is $|a|$, and the period is $\frac{360°}{|b|}$ or $\frac{2\pi}{|b|}$.
- The period of $y = a \tan b\theta$ is $\frac{180°}{|b|}$ or $\frac{\pi}{|b|}$.

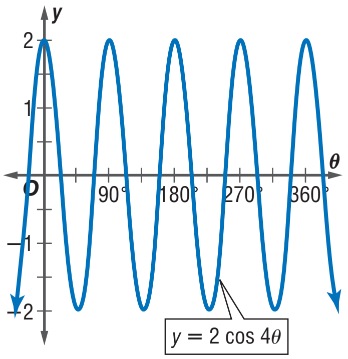
Example

Find the amplitude and period of $y = 2 \cos 4\theta$. Then graph the function.

The amplitude is $|2|$ or 2.

The period is $\frac{360°}{|4|}$ or 90°.

Use the amplitude and period to graph the function.



Exercises **Find the amplitude, if it exists, and period of each function. Then graph each function.** *See Example 1 on page 765.*

9. $y = -\frac{1}{2} \cos \theta$
10. $y = 4 \sin 2\theta$
11. $y = \sin \frac{1}{2}\theta$
12. $y = 5 \sec \theta$
13. $y = \frac{1}{2} \csc \frac{2}{3}\theta$
14. $y = \tan 4\theta$

CONTENTS

14-2 Translations of Trigonometric Graphs

See pages 769–776.

Concept Summary

- For trigonometric functions of the form $y = a \sin b(\theta - h)$, $y = a \cos (\theta - h)$, and $y = a \tan (\theta - h)$, the phase shift is to the right when $h > 0$ and to the left when $h < 0$.
- For trigonometric functions of the form $y = a \sin b(\theta - h) + k$, $y = a \cos (\theta - h) + k$, and $y = a \tan (\theta - h) + k$, the vertical shift is up when $k > 0$ and down when $k < 0$.

Example **State the vertical shift, amplitude, period, and phase shift of $y = 3 \sin \left[2\left(\theta - \frac{\pi}{2}\right)\right] - 2$. Then graph the function.**

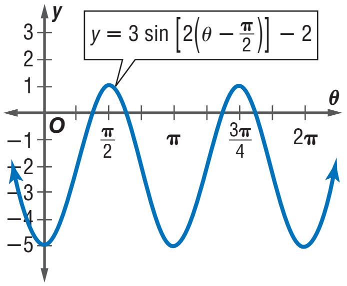
Identify the values of k, a, b, and h.

$k = -2$, so the vertical shift is –2.

$a = 3$, so the amplitude is 3.

$b = 2$, so the period is $\frac{2\pi}{|2|}$ or π.

$h = \frac{\pi}{2}$, so the phase shift is $\frac{\pi}{2}$ to the right.



Exercises **State the vertical shift, amplitude, period, and phase shift of each function. Then graph the function.** *See Example 3 on page 772.*

15. $y = \frac{1}{2} \sin [2(\theta - 60°)] - 1$

16. $y = 2 \tan \left[\frac{1}{4}(\theta - 90°)\right] + 3$

17. $y = 3 \sec \left[\frac{1}{2}\left(\theta + \frac{\pi}{4}\right)\right] + 1$

18. $y = \frac{1}{3} \cos \left[\frac{1}{3}\left(\theta - \frac{2\pi}{3}\right)\right] - 2$

14-3 Trigonometric Identities

See pages 777–781.

Concept Summary

- Quotient Identities: $\tan \theta = \frac{\sin \theta}{\cos \theta}$, $\cot \theta = \frac{\cos \theta}{\sin \theta}$
- Reciprocal Identities: $\csc \theta = \frac{1}{\sin \theta}$, $\sec \theta = \frac{1}{\cos \theta}$, $\cot \theta = \frac{1}{\tan \theta}$
- Pythagorean Identities: $\cos^2 \theta + \sin^2 \theta = 1$, $\tan^2 \theta + 1 = \sec^2 \theta$, $\cot^2 \theta + 1 = \sec^2 \theta$

Example **Simplify $\sin \theta \cot \theta \cos \theta$.**

$$\sin \theta \cot \theta \cos \theta = \frac{\sin \theta}{1} \cdot \frac{\cos \theta}{\sin \theta} \cdot \frac{\cos \theta}{1} \qquad \cot \theta = \frac{\cos \theta}{\sin \theta}$$

$$= \cos^2 \theta \qquad \text{Multiply.}$$

Exercises **Find the value of each expression.** *See Example 1 on page 778.*

19. $\cot \theta$, if $\csc \theta = -\frac{5}{3}$; $270° < \theta < 360°$

20. $\sec \theta$, if $\sin \theta = \frac{1}{2}$; $0° \leq \theta < 90°$

Simplify each expression. *See Example 2 on page 778.*

21. $\sin \theta \csc \theta - \cos^2 \theta$

22. $\cos^2 \theta \sec \theta \csc \theta$

23. $\cos \theta + \sin \theta \tan \theta$

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14-4 Verifying Trigonometric Identities

See pages 782–785.

Concept Summary

- Use the basic trigonometric identities to transform one or both sides of a trigonometric equation into the same form.

Example **Verify that $\tan \theta + \cot \theta = \sec \theta \csc \theta$.**

$\tan \theta + \cot \theta \overset{?}{=} \sec \theta \csc \theta$ Original equation

$\frac{\sin \theta}{\cos \theta} + \frac{\cos \theta}{\sin \theta} \overset{?}{=} \sec \theta \csc \theta$ $\tan \theta = \frac{\sin \theta}{\cos \theta}$, $\cot \theta = \frac{\cos \theta}{\sin \theta}$

$\frac{\sin^2 \theta + \cos^2 \theta}{\cos \theta \sin \theta} \overset{?}{=} \sec \theta \csc \theta$ Rewrite using the LCD, $\cos \theta \sin \theta$.

$\frac{1}{\cos \theta \sin \theta} \overset{?}{=} \sec \theta \csc \theta$ $\sin^2 \theta + \cos^2 \theta = 1$

$\frac{1}{\cos \theta} \cdot \frac{1}{\sin \theta} \overset{?}{=} \sec \theta \csc \theta$ Rewrite as the product of two expressions.

$\sec \theta \csc \theta = \sec \theta \csc \theta$ $\frac{1}{\cos \theta} = \sec \theta$, $\frac{1}{\sin \theta} = \csc \theta$

Exercises **Verify that each of the following is an identity.**
See Examples 1–3 on pages 782–783.

24. $\frac{\sin \theta}{\tan \theta} + \frac{\cos \theta}{\cot \theta} = \cos \theta + \sin \theta$

25. $\frac{\sin \theta}{1 - \cos \theta} = \csc \theta + \cot \theta$

26. $\cot^2 \theta \sec^2 \theta = 1 + \cot^2 \theta$

27. $\sec \theta \, (\sec \theta - \cos \theta) = \tan^2 \theta$

14-5 Sum and Difference of Angles Formulas

See pages 786–790.

Concept Summary

- For all values of α and β : $\cos(\alpha \pm \beta) = \cos \alpha \cos \beta \mp \sin \alpha \sin \beta$
  $\sin(\alpha \pm \beta) = \sin \alpha \cos \beta \pm \cos \alpha \sin \beta$

Example **Find the exact value of sin 195°.**

$\sin 195° = \sin(150° + 45°)$ $195° = 150° + 45°$

$= \sin 150° \cos 45° + \cos 150° \sin 45°$ $\alpha = 150°, \beta = 45°$

$= \left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right) + \left(-\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)$ Evaluate each expression.

$= \frac{\sqrt{2} - \sqrt{6}}{4}$ Simplify.

Exercises **Find the exact value of each expression.** *See Example 1 on page 787.*

28. cos 15°
29. cos 285°
30. sin 150°
31. sin 195°
32. cos (−210°)
33. sin (−105°)

Verify that each of the following is an identity. *See Example 3 on page 788.*

34. $\cos(90° + \theta) = -\sin \theta$

35. $\sin(30° - \theta) = \cos(60° + \theta)$

36. $\sin(\theta + \pi) = -\sin \theta$

37. $-\cos \theta = \cos(\pi + \theta)$

- Extra Practice, see pages 859–861.
- Mixed Problem Solving, see page 875.

14-6 Double-Angle and Half-Angle Formulas

See pages 791–797.

Concept Summary

- Double-angle formulas: $\sin 2\theta = 2\sin\theta\cos\theta$, $\cos 2\theta = \cos^2\theta - \sin^2\theta$, $\cos 2\theta = 1 - 2\sin^2\theta$, $\cos 2\theta = 2\cos^2\theta - 1$
- Half-angle formulas: $\sin\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{2}}$, $\cos\frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\alpha}{2}}$

Example **Verify that $\csc 2\theta = \frac{\sec\theta}{2\sin\theta}$ is an identity.**

$\csc 2\theta \overset{?}{=} \frac{\sec\theta}{2\sin\theta}$ Original equation

$\frac{1}{\sin 2\theta} \overset{?}{=} \frac{\frac{1}{\cos\theta}}{2\sin\theta}$ $\csc\theta = \frac{1}{\sin\theta}$, $\sec\theta = \frac{1}{\cos\theta}$

$\frac{1}{\sin 2\theta} \overset{?}{=} \frac{1}{2\sin\theta\cos\theta}$ Simplify the complex fraction.

$\frac{1}{\sin 2\theta} = \frac{1}{\sin 2\theta}$ $2\sin\theta\cos\theta = \sin 2\theta$

Exercises **Find the exact values of $\sin 2\theta$, $\cos 2\theta$, $\sin\frac{\theta}{2}$, and $\cos\frac{\theta}{2}$ for each of the following.** *See Examples 1 and 2 on pages 792 and 793.*

38. $\sin\theta = \frac{1}{4}$; $0° < \theta < 90°$

39. $\sin\theta = -\frac{5}{13}$; $180° < \theta < 270°$

40. $\cos\theta = -\frac{5}{17}$; $90° < \theta < 180°$

41. $\cos\theta = \frac{12}{13}$; $270° < \theta < 360°$

14-7 Solving Trigonometric Equations

See pages 799–804.

Concept Summary

- Solve trigonometric equations by factoring or by using trigonometric identities.

Example **Solve $\sin 2\theta + \sin\theta = 0$ if $0° \leq \theta < 360°$.**

$\sin 2\theta + \sin\theta = 0$ Original equation

$2\sin\theta\cos\theta + \sin\theta = 0$ $\sin 2\theta = 2\sin\theta\cos\theta$

$\sin\theta(2\cos\theta + 1) = 0$ Factor.

$\sin\theta = 0$ or $2\cos\theta + 1 = 0$

$\theta = 0°$ or $180°$ $\theta = 120°$ or $240°$

Exercises **Find all solutions of each equation for the interval $0° \leq \theta < 360°$.** *See Example 1 on page 799.*

42. $2\sin 2\theta = 1$

43. $2\cos^2\theta + \sin^2\theta = 2\cos\theta$

Solve each equation for all values of θ if θ is measured in radians. *See Example 2 on page 800.*

44. $6\sin^2\theta - 5\sin\theta - 4 = 0$

45. $2\cos^2\theta = 3\sin\theta$

Chapter 14 Practice Test

Vocabulary and Concepts

Choose the correct term to complete each sentence.

1. The (*period*, *phase shift*) of $y = 3 \sin 2(\theta - 60°) + 2$ is 120°.

2. A midline is used with a (*phase shift*, *vertical shift*) of a trigonometric function.

3. The amplitude of $y = \frac{1}{3} \cos [3(\theta + 4)] - 1$ is $\left(\frac{1}{3}, 3\right)$.

4. The (*cosine*, *cosecant*) has no amplitude.

Skills and Applications

State the vertical shift, amplitude, period, and phase shift of each function. Then graph the function.

5. $y = \frac{2}{3} \sin 2\theta + 5$

6. $y = 4 \cos \left[\frac{1}{2}(\theta + 30°)\right] - 1$

Find the value of each expression.

7. $\tan \theta$, if $\sin \theta = \frac{1}{2}$; $90° < \theta < 180°$

8. $\sec \theta$, if $\cot \theta = \frac{3}{4}$; $180° < \theta < 270°$

Verify that each of the following is an identity.

9. $(\sin \theta - \cos \theta)^2 = 1 - \sin 2\theta$

10. $\frac{\cos \theta}{1 - \sin^2 \theta} = \sec \theta$

11. $\frac{\sec \theta}{\sin \theta} - \frac{\sin \theta}{\cos \theta} = \cot \theta$

12. $\frac{1 + \tan^2 \theta}{\cos^2 \theta} = \sec^4 \theta$

Find the exact value of each expression.

13. $\cos 285°$

14. $\sin 345°$

15. $\sin (-225°)$

16. $\cos 480°$

17. $\cos 67.5°$

18. $\sin 75°$

Solve each equation for all values of θ if θ is measured in degrees.

19. $\sec \theta = 1 + \tan \theta$

20. $\cos 2\theta = \cos \theta$

21. $\cos 2\theta + \sin \theta = 1$

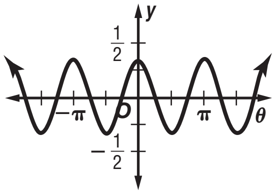
22. $\sin \theta = \tan \theta$

GOLF **For Exercises 23 and 24, use the following information.**
A golf ball is hit with an initial velocity of 100 feet per second. The distance the ball travels is found by the formula $d = \frac{{v_0}^2}{g} \sin 2\theta$, where v_0 is the initial velocity, g is the acceleration due to gravity, 32 feet per second squared, and θ is the measurement of the angle that the path of the ball makes with the ground.

23. Find the distance that the ball travels if the angle between the path of the ball and the ground measures 60°.

24. If a ball travels 312.5 feet, what was the angle the path of the ball made with the ground to the nearest degree?

25. **STANDARDIZED TEST PRACTICE** Identify the equation of the graphed function.



Ⓐ $y = 3 \cos 2\theta$

Ⓑ $y = \frac{1}{3} \cos 2\theta$

Ⓒ $y = 3 \cos \frac{1}{2}\theta$

Ⓓ $y = \frac{1}{3} \cos \frac{1}{2}\theta$

www.algebra2.com/chapter_test

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Chapter 14 Standardized Test Practice

Part 1 Multiple Choice

Record your answers on the answer sheet provided by your teacher or on a sheet of paper.

1. Which of the following is *not* equal to 3.5×10^{-2}?

Ⓐ $\frac{35}{1000}$
Ⓑ 0.035
Ⓒ $\frac{7}{200}$
Ⓓ (0.5)(0.007)

2. The sum of five consecutive odd integers is 55. What is the sum of the greatest and least of these integers?

Ⓐ 11
Ⓑ 22
Ⓒ 26
Ⓓ 30

3. If 8 bananas cost a cents and 6 oranges cost b cents, what is the cost of 2 bananas and 2 oranges in terms of a and b?

Ⓐ $\frac{ab}{12}$
Ⓑ $3a + \frac{b}{3}$
Ⓒ $3a + 4b$
Ⓓ $\frac{3a + 4b}{12}$

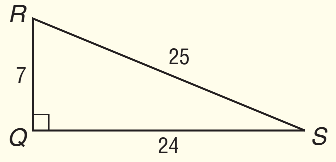
4. A bag contains 16 peppermint candies, 10 butterscotch candies, and 8 cherry candies. Emma chooses one piece at random, puts it in her pocket, and then repeats the process. If she has chosen 3 peppermint candies, 2 butterscotch candies, and 1 cherry candy, what is the probability that the next piece of candy she chooses will be cherry?

Ⓐ $\frac{7}{34}$
Ⓑ $\frac{8}{34}$
Ⓒ $\frac{1}{4}$
Ⓓ $\frac{3}{4}$

5. What is the value of $\dfrac{\sin \frac{\pi}{6}}{\cos \frac{2\pi}{3}}$?

Ⓐ $-\sqrt{3}$
Ⓑ -1
Ⓒ $-\frac{\sqrt{3}}{3}$
Ⓓ 1

6. In right triangle QRS, what is the value of $\tan R$?



Ⓐ $\frac{7}{25}$
Ⓑ $\frac{7}{24}$
Ⓒ $\frac{25}{24}$
Ⓓ $\frac{24}{7}$

7. What is the value of $\sin\left(\cos^{-1} \frac{1}{3}\right)$?

Ⓐ $\frac{2}{3}$
Ⓑ $\frac{2\sqrt{2}}{3}$
Ⓒ $\frac{\sqrt{2}}{3}$
Ⓓ $\frac{\sqrt{6}}{3}$

8. What is the least positive value for x where $y = \sin 2x$ reaches its minimum?

Ⓐ $\frac{\pi}{2}$
Ⓑ π
Ⓒ $\frac{3\pi}{4}$
Ⓓ $\frac{3\pi}{2}$

9. Which of the following is equivalent to $\frac{\sin^2 \theta + \cos^2 \theta}{\sec^2 \theta}$?

Ⓐ $\cos^2 \theta$
Ⓑ $\sin^2 \theta$
Ⓒ $\tan^2 \theta$
Ⓓ $\sin^2 \theta + 1$

10. If $\cos \theta = -\frac{1}{2}$ and θ is in Quadrant II, what is the value of $\sin 2\theta$?

Ⓐ $\frac{1}{2}$
Ⓑ $-\frac{1}{2}$
Ⓒ $\frac{\sqrt{3}}{2}$
Ⓓ $-\frac{\sqrt{3}}{2}$

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Part 2 Short Response/Grid In

Record your answers on the answer sheet provided by your teacher or on a sheet of paper.

11. If k is a positive integer, and $7k + 3$ equals a prime number that is less than 50, then what is one possible value of $7k + 3$?

12. It costs $8 to make a book. The selling price will include an additional 200%. What will be the selling price?

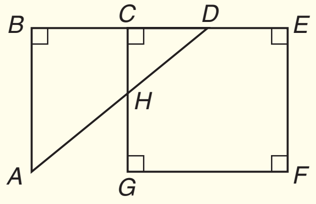
13. The mean of seven numbers is 0. The sum of three of the numbers is -9. What is the sum of the remaining four numbers?

14. If $4a - 6b = 0$ and $c = 9b$, what is the ratio of a to c?

15. What is the value of x if $\frac{3^3 \cdot 3}{\sqrt{81}} = 3^x$?

16. The ages of children at a party are 6, 7, 6, 6, 7, 7, 8, 6, 7, 8, 9, 7, and 7. Let N represent the median of their ages and m represent the mode. What is $N - m$?

17. In the figure below, $CEFG$ is a square, ABD is a right triangle, D is the midpoint of side CE, H is the midpoint of side CG, and C is the midpoint of side BD. $BCDE$ is a line segment, and AHD is a line segment. If the measure of the area of square $CEFG$ is 16, what is the measure of the area of quadrilateral $ABCH$?



18. A line with a slope of $\frac{3}{8}$ passes through points $(6, 4n)$ and $(0, n)$. What is the value of n?

19. If $\sin 60° = \frac{\sqrt{3}}{2}$, what is the value of $\sin^2 30° + \cos^2 30°$?

Part 3 Quantitive Comparison

Compare the quantity in Column A and the quantity in Column B. Then determine whether:

- Ⓐ **the quantity in Column A is greater,**
- Ⓑ **the quantity in Column B is greater,**
- Ⓒ **the two quantities are equal, or**
- Ⓓ **the relationship cannot be determined from the information given.**

	Column A	Column B
20.	the length of a diagonal of a square whose area is 100	the length of a diagonal of a 6 × 8 rectangle

21. $\left\langle c \right\rangle = c^2 + \frac{2}{c} - 2$

$\left\langle 1 \right\rangle$	$\left\langle 2 \right\rangle$

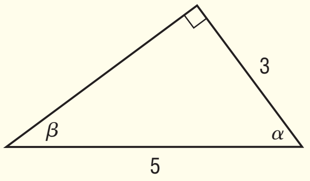
22. $w = 2x,\ x = \frac{1}{2}w$

w	x

23. $(a + b)^2 = a^2 + b^2$

$(a - b)^2$	$a^2 + b^2$

24.



$\tan \beta$	$\sin \alpha$

The Princeton Review

Test-Taking Tip

Always write down your calculations on scrap paper or in the test booklet, even if you think you can do the calculations in your head. Writing down your calculations will help you avoid making simple mistakes.

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