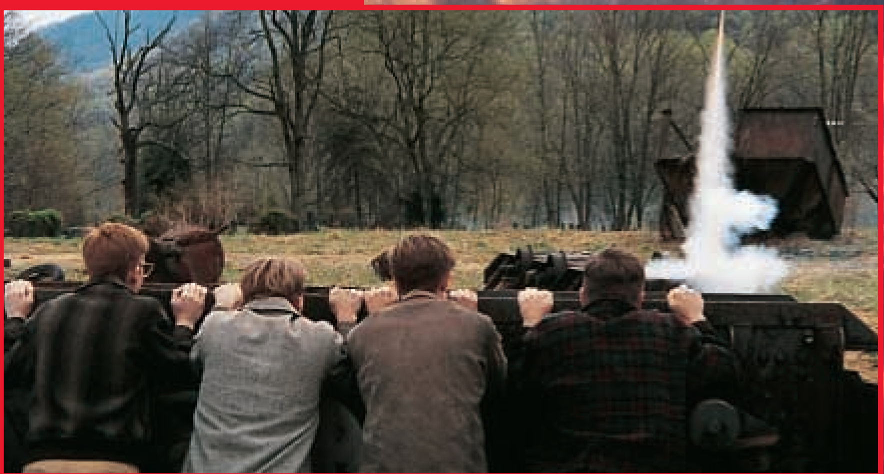
UNIT 5

Trigonometry

Trigonometry is used in navigation, physics, and construction, among other fields. In this unit, you will learn about trigonometric functions, graphs, and identities.



CONTENTS

WebQuest Internet Project

Trig Class Angles for Lessons in Lit

Source: *USA TODAY,* November 21, 2000

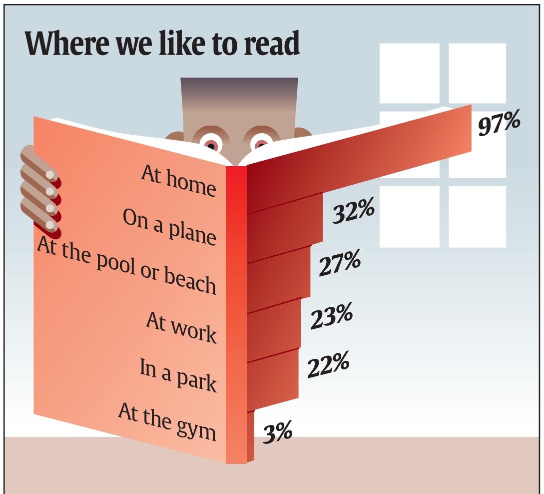
"The groans from the trigonometry students immediately told teacher Michael Buchanan what the class thought of his idea to read Homer Hickam's *October Sky*. In the story, in order to accomplish what they would like, the kids had to teach themselves trig, calculus, and physics." In this project, you will research applications of trigonometry as it applies to a possible career for you.

Log on to www.algebra2.com/webquest. Begin your WebQuest by reading the Task.

Then continue working on your WebQuest as you study Unit 5.

Lesson	13-1	14-2
Page	708	775

USA TODAY Snapshots®



Source: Buskin/Audits & Survey Worldwide for *American Demographics*

By Cindy Hall and Sam Ward, USA TODAY

Chapter 13

Trigonometric Functions

What You'll Learn

- **Lessons 13-1, 13-2, 13-3, 13-6, and 13-7** Find values of trigonometric functions.
- **Lessons 13-1, 13-4, and 13-5** Solve problems by using right triangle trigonometry.
- **Lessons 13-4 and 13-5** Solve triangles by using the Law of Sines and Law of Cosines.

Key Vocabulary

- solve a right triangle (p. 704)
- radian (p. 710)
- Law of Sines (p. 726)
- Law of Cosines (p. 733)
- circular function (p. 740)

Why It's Important

Trigonometry is the study of the relationships among the angles and sides of right triangles. One of the many real-world applications of trigonometric functions involves solving problems using indirect measurement. For example, surveyors use a trigonometric function to find the heights of buildings. *You will learn how architects who design fountains use a trigonometric function to aim the water jets in Lesson 13-7.*

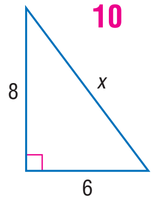
Getting Started

Prerequisite Skills To be successful in this chapter, you'll need to master these skills and be able to apply them in problem-solving situations. Review these skills before beginning Chapter 13.

For Lessons 13-1 and 13-3 — Pythagorean Theorem

Find the value of x to the nearest tenth. *(For review, see pages 820 and 821.)*

1. **10**



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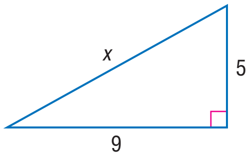
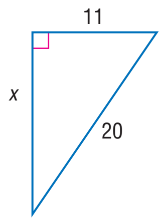


10.3

3. **16.7**



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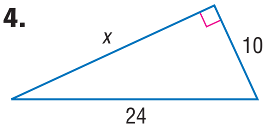
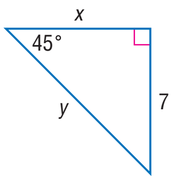


21.8

For Lesson 13-1 — 45°-45°-90° and 30°-60°-90° Triangles

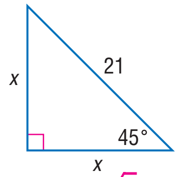
Find each missing measure. Write all radicals in simplest form.

5.



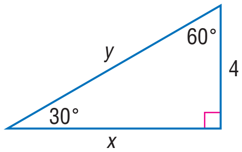
$x = 7,\ y = 7\sqrt{2}$

6.



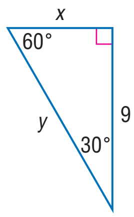
$x = \dfrac{21\sqrt{2}}{2}$

7.



$x = 4\sqrt{3},\ y = 8$

8.



$x = 3\sqrt{3},\ y = 6\sqrt{3}$

For Lesson 13-7 — Inverse Functions

Find the inverse of each function. Then graph the function and its inverse.
(For review, see Lesson 7-8.) **9–12. See pp. 759A–759D for graphs.**

9. $f(x) = x + 3$ $f^{-1}(x) = x - 3$

10. $f(x) = \frac{x-2}{5}$ $f^{-1}(x) = 5x + 2$

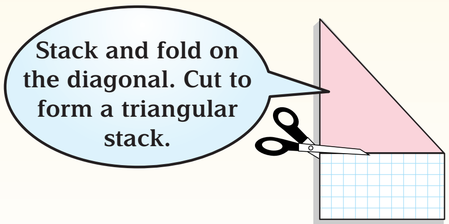
11. $f(x) = x^2 - 4$ $f^{-1}(x) = \pm\sqrt{x+4}$

12. $f(x) = -7x - 9$ $f^{-1}(x) = \frac{-x-9}{7}$

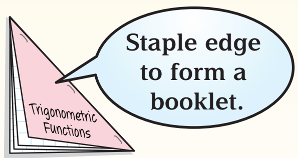
FOLDABLES™ Study Organizer

Make this Foldable to help you organize information about trigonometric functions. Begin with one sheet of construction paper and two pieces of grid paper.

Step 1 Fold and Cut



Step 2 Staple and Label



Reading and Writing As you read and study the chapter, you can write notes, draw diagrams, and record formulas on the grid paper.

Spreadsheet Investigation

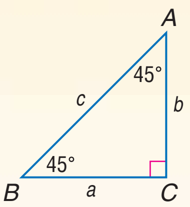
A Preview of Lesson 13-1

Special Right Triangles

You can use a computer spreadsheet program to investigate the relationships among the ratios of the side measures of special right triangles.

Example

The legs of a 45°-45°-90° triangle, *a* and *b*, are equal in measure. Use a spreadsheet to investigate the dimensions of 45°-45°-90° triangles. What patterns do you observe in the ratios of the side measures of these triangles?



=SQRT(A2^2 + B2^2)　=B2/A2　=B2/C2　=A2/C2

45-45-90 Triangles

	A	B	C	D	E	F
1	a	b	c	b/a	b/c	a/c
2	1	1	1.41421356	1	0.70710678	0.70710678
3	2	2	2.82842712	1	0.70710678	0.70710678
4	3	3	4.24264069	1	0.70710678	0.70710678
5	4	4	5.65685425	1	0.70710678	0.70710678
6	5	5	7.07106781	1	0.70710678	0.70710678

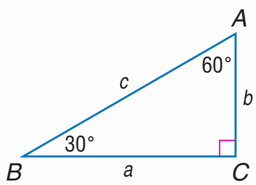
Sheet1 / Sheet2 / S

The spreadsheet shows the formula that will calculate the length of side *c*. The formula uses the Pythagorean Theorem in the form $c = \sqrt{a^2 + b^2}$. Since 45°-45°-90° triangles share the same angle measures, the triangles listed in the spreadsheet are all similar triangles. Notice that all of the ratios of side *b* to side *a* are 1. All of the ratios of side *b* to side *c* and of side *a* to side *c* are approximately 0.71.

Exercises

For Exercises 1 and 2, use the spreadsheet below for 30°-60°-90° triangles.

If the measure of one leg of a right triangle and the measure of the hypotenuse are in ratio of 1 to 2, then the acute angles of the triangle measure 30° and 60°.



30-60-90 Triangles

	A	B	C	D	E	F
1	a	b	c	b/a	b/c	a/c
2	1		2			
3	2		4			
4	3		6			
5	4		8			
6	5		10			

Sheet1 / Sheet2 / Sheet3

1. Copy and complete the spreadsheet above.
2. Describe the relationship among the 30°-60°-90° triangles whose dimensions are given.
3. What patterns do you observe in the ratios of the side measures of these triangles?

13-1 Right Triangle Trigonometry

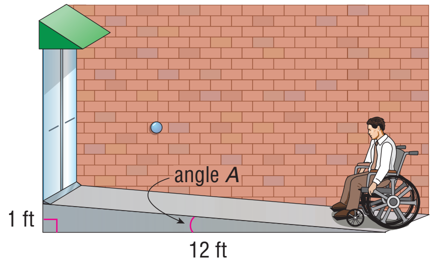
What You'll Learn

- Find values of trigonometric functions for acute angles.
- Solve problems involving right triangles.

How is trigonometry used in building construction?

The Americans with Disabilities Act (ADA) provides regulations designed to make public buildings accessible to all. Under this act, the slope of an entrance ramp designed for those with mobility disabilities must not exceed a ratio of 1 to 12. This means that for every 12 units of horizontal run, the ramp can rise or fall no more than 1 unit.

When viewed from the side, a ramp forms a right triangle. The slope of the ramp can be described by the *tangent* of the angle the ramp makes with the ground. In this example, the tangent of angle A is $\frac{1}{12}$.



Vocabulary

- trigonometry
- trigonometric functions
- sine
- cosine
- tangent
- cosecant
- secant
- cotangent
- solve a right triangle
- angle of elevation
- angle of depression

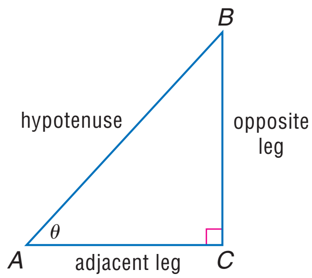
Study Tip

Reading Math

The word *trigonometry* is derived from two Greek words—*trigon* meaning triangle and *metra* meaning measurement.

TRIGONOMETRIC VALUES The tangent of an angle is one of the ratios used in trigonometry. **Trigonometry** is the study of the relationships among the angles and sides of a right triangle.

Consider right triangle ABC in which the measure of acute angle A is identified by the Greek letter *theta*, θ. The sides of the triangle are the *hypotenuse*, the *leg opposite* θ, and the *leg adjacent to* θ.



Using these sides, you can define six **trigonometric functions**: **sine**, **cosine**, **tangent**, **cosecant**, **secant**, and **cotangent**. These functions are abbreviated sin, cos, tan, sec, csc, and cot, respectively.

Key Concept — *Trigonometric Functions*

If θ is the measure of an acute angle of a right triangle, *opp* is the measure of the leg opposite θ, *adj* is the measure of the leg adjacent to θ, and *hyp* is the measure of the hypotenuse, then the following are true.

$$\sin\theta = \frac{\text{opp}}{\text{hyp}} \qquad \cos\theta = \frac{\text{adj}}{\text{hyp}} \qquad \tan\theta = \frac{\text{opp}}{\text{adj}}$$

$$\csc\theta = \frac{\text{hyp}}{\text{opp}} \qquad \sec\theta = \frac{\text{hyp}}{\text{adj}} \qquad \cot\theta = \frac{\text{adj}}{\text{opp}}$$

Notice that the sine, cosine, and tangent functions are reciprocals of the cosecant, secant, and cotangent functions, respectively. Thus, the following are also true.

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

Study Tip

Memorize Trigonometric Ratios

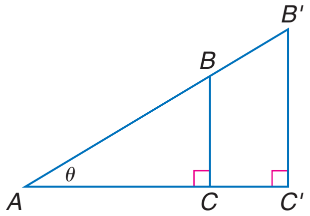
SOH-CAH-TOA is a mnemonic device for remembering the first letter of each word in the ratios for sine, cosine, and tangent.

$\sin \theta = \frac{\text{opp}}{\text{hyp}}$

$\cos \theta = \frac{\text{adj}}{\text{hyp}}$

$\tan \theta = \frac{\text{opp}}{\text{adj}}$

The domain of each of these trigonometric functions is the set of all acute angles θ of a right triangle. The values of the functions depend only on the measure of θ and not on the size of the right triangle. For example, consider $\sin \theta$ in the figure at the right.



Using $\triangle ABC$:

$$\sin \theta = \frac{BC}{AB}$$

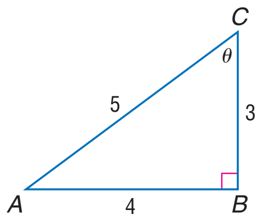
Using $\triangle AB'C'$:

$$\sin \theta = \frac{B'C'}{AB'}$$

The right triangles are similar because they share angle θ. Since they are similar, the ratios of corresponding sides are equal. That is, $\frac{BC}{AB} = \frac{B'C'}{AB'}$. Therefore, you will find the same value for $\sin \theta$ regardless of which triangle you use.

Example 1 Find Trigonometric Values

Find the values of the six trigonometric functions for angle θ.



For this triangle, the leg opposite θ is $\overline{AB}$, and the leg adjacent to θ is $\overline{CB}$. Recall that the hypotenuse is always the longest side of a right triangle, in this case $\overline{AC}$.

Use opp = 4, adj = 3, and hyp = 5 to write each trigonometric ratio.

$\sin \theta = \frac{\text{opp}}{\text{hyp}} = \frac{4}{5}$ $\quad$ $\cos \theta = \frac{\text{adj}}{\text{hyp}} = \frac{3}{5}$ $\quad$ $\tan \theta = \frac{\text{opp}}{\text{adj}} = \frac{4}{3}$

$\csc \theta = \frac{\text{hyp}}{\text{opp}} = \frac{5}{4}$ $\quad$ $\sec \theta = \frac{\text{hyp}}{\text{adj}} = \frac{5}{3}$ $\quad$ $\cot \theta = \frac{\text{adj}}{\text{opp}} = \frac{3}{4}$

Throughout Unit 5, a capital letter will be used to denote both a vertex of a triangle and the measure of the angle at that vertex. The same letter in lowercase will be used to denote the side opposite that angle and its measure.

Standardized Test Practice

A B C D

Example 2 Use One Trigonometric Ratio to Find Another

Multiple-Choice Test Item

If $\cos A = \frac{2}{5}$, find the value of $\tan A$.

(A) $\frac{5}{2}$ $\quad$ (B) $\frac{2\sqrt{21}}{21}$ $\quad$ (C) $\frac{\sqrt{21}}{2}$ $\quad$ (D) $\sqrt{21}$

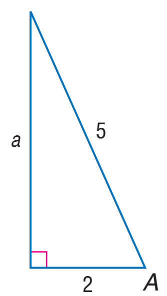
The Princeton Review

Test-Taking Tip

Whenever necessary or helpful, draw a diagram of the situation.

Read the Test Item

Begin by drawing a right triangle and labeling one acute angle A. Since $\cos \theta = \frac{\text{adj}}{\text{hyp}}$ and $\cos A = \frac{2}{5}$ in this case, label the adjacent leg 2 and the hypotenuse 5.



Solve the Test Item

Use the Pythagorean Theorem to find a.

$a^2 + b^2 = c^2$	Pythagorean Theorem
$a^2 + 2^2 = 5^2$	Replace b with 2 and c with 5.
$a^2 + 4 = 25$	Simplify.
$a^2 = 21$	Subtract 4 from each side.
$a = \sqrt{21}$	Take the square root of each side.

Now find tan A.

$\tan A = \frac{\text{opp}}{\text{adj}}$ Tangent ratio

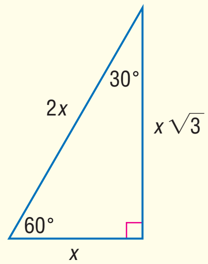
$= \frac{\sqrt{21}}{2}$ Replace *opp* with $\sqrt{21}$ and *adj* with 2.

The answer is C.

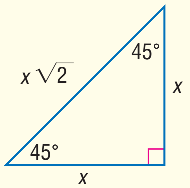
Angles that measure 30°, 45°, and 60° occur frequently in trigonometry. The table below gives the values of the six trigonometric functions for these angles. To remember these values, use the properties of 30°-60°-90° and 45°-45°-90° triangles.

Key Concept — Trigonometric Values for Special Angles

30°-60°-90° Triangle



45°-45°-90° Triangle



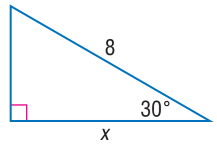
θ	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\csc\theta$	$\sec\theta$	$\cot\theta$
30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$	$\sqrt{3}$
45°	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	$\sqrt{2}$	$\sqrt{2}$	1
60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2	$\frac{\sqrt{3}}{3}$

You will verify some of these values in Exercises 27 and 28.

RIGHT TRIANGLE PROBLEMS You can use trigonometric functions to solve problems involving right triangles.

Example 3 Find a Missing Side Length of a Right Triangle

Write an equation involving sin, cos, or tan that can be used to find the value of *x*. Then solve the equation. Round to the nearest tenth.



The measure of the hypotenuse is 8. The side with the missing length is *adjacent* to the angle measuring 30°. The trigonometric function relating the adjacent side of a right triangle and the hypotenuse is the cosine function.

$\cos\theta = \frac{\text{adj}}{\text{hyp}}$ cosine ratio

$\cos 30° = \frac{x}{8}$ Replace θ with 30°, *adj* with *x*, and *hyp* with 8.

$\frac{\sqrt{3}}{2} = \frac{x}{8}$ $\cos 30° = \frac{\sqrt{3}}{2}$.

$4\sqrt{3} = x$ Multiply each side by 8.

The value of x is $4\sqrt{3}$ or about 6.9.

Study Tip

Common Misconception
The $\cos^{-1} x$ on a graphing calculator does *not* find $\frac{1}{\cos x}$. To find sec x or $\frac{1}{\cos x}$, find cos x and then use the $\boxed{x^{-1}}$ key.

A calculator can be used to find the value of trigonometric functions for *any* angle, not just the special angles mentioned. Use $\boxed{\text{SIN}}$, $\boxed{\text{COS}}$, and $\boxed{\text{TAN}}$ for sine, cosine, and tangent. Use these keys and the reciprocal key, $\boxed{x^{-1}}$, for cosecant, secant, and cotangent. Be sure your calculator is in degree mode.

Here are some calculator examples.

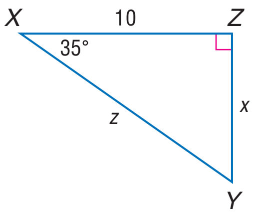
cos 46° **KEYSTROKES:** [COS] 46 [ENTER] .6946583705

cot 20° **KEYSTROKES:** [TAN] 20 [ENTER] [x^{-1}] [ENTER] 2.747477419

If you know the measures of any two sides of a right triangle or the measures of one side and one acute angle, you can determine the measures of all the sides and angles of the triangle. This process of finding the missing measures is known as **solving a right triangle**.

Example 4 Solve a Right Triangle

Solve $\triangle XYZ$. Round measures of sides to the nearest tenth and measures of angles to the nearest degree.



You know the measures of one side, one acute angle, and the right angle. You need to find x, z, and Y.

Find x and z.

$$\tan 35° = \frac{x}{10} \qquad \sec 35° = \frac{z}{10}$$

$$10 \tan 35° = x \qquad \frac{1}{\cos 35°} = \frac{z}{10}$$

$$7.0 \approx x \qquad \frac{10}{\cos 35°} = z$$

$$12.2 \approx z$$

Find Y.

$35° + Y = 90°$ Angles X and Y are complementary.

$Y = 55°$ Solve for Y.

Therefore, $Y = 55°$, $x \approx 7.0$, and $z \approx 12.2$.

Study Tip

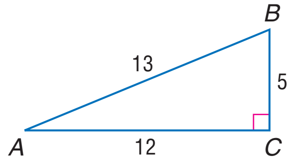
Error in Measurement

The value of z in Example 4 is found using the secant instead of using the Pythagorean Theorem. This is because the secant uses values given in the problem rather than calculated values.

Use the inverse capabilities of your calculator to find the measure of an angle when one of its trigonometric ratios is known. For example, use the SIN^{-1} function to find the measure of an angle when the sine of the angle is known. *You will learn more about inverses of trigonometric functions in Lesson 13-7.*

Example 5 Find Missing Angle Measures of Right Triangles

Solve $\triangle ABC$. Round measures of sides to the nearest tenth and measures of angles to the nearest degree.



You know the measures of the sides. You need to find A and B.

Find A. $\sin A = \frac{5}{13}$ $\sin A = \frac{\text{opp}}{\text{hyp}}$

Use a calculator and the SIN^{-1} function to find the angle whose sine is $\frac{5}{13}$.

KEYSTROKES: [2nd] [SIN^{-1}] 5 [÷] 13 [)] [ENTER] 22.61986495

To the nearest degree, $A \approx 23°$.

Find B. $23° + B \approx 90°$ Angles A and B are complementary.

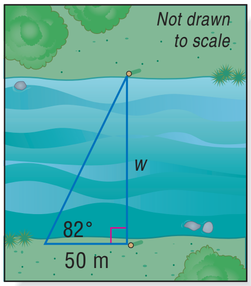
$B \approx 67°$ Solve for B.

Therefore, $A \approx 23°$ and $B \approx 67°$.

Trigonometry has many practical applications. Among the most important is the ability to find distances or lengths that either cannot be measured directly or are not easily measured directly.

Example 6 Indirect Measurement

BRIDGE CONSTRUCTION **In order to construct a bridge across a river, the width of the river at that location must be determined. Suppose a stake is planted on one side of the river directly across from a second stake on the opposite side. At a distance 50 meters to the left of the stake, an angle of 82° is measured between the two stakes. Find the width of the river.**



Let w represent the width of the river at that location. Write an equation using a trigonometric function that involves the ratio of the distance w and 50.

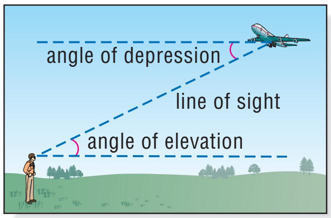
$\tan 82° = \frac{w}{50}$ $\tan \theta = \frac{\text{opp}}{\text{adj}}$

$50 \tan 82° = w$ Multiply each side by 50.

$355.8 \approx w$ Use a calculator.

The width of the river is about 355.8 meters.

Some applications of trigonometry use an angle of elevation or depression. In the figure at the right, the angle formed by the line of sight from the observer and a line parallel to the ground is called the **angle of elevation**. The angle formed by the line of sight from the plane and a line parallel to the ground is called the **angle of depression**.



The angle of elevation and the angle of depression are congruent since they are alternate interior angles of parallel lines.

More About. . .

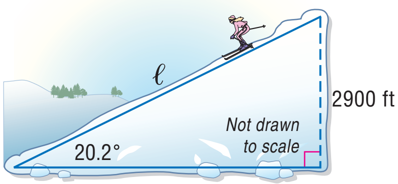
Skiing

The average annual snowfall in Snowbird, Utah, is 500 inches. The longest designated run there is Chip's Run, at 2.5 miles.

Source: www.utahskiing.com

Example 7 Use an Angle of Elevation

SKIING **The Aerial run in Snowbird, Utah, has an angle of elevation of 20.2°. Its vertical drop is 2900 feet. Estimate the length of this run.**



Let ℓ represent the length of the run. Write an equation using a trigonometric function that involves the ratio of ℓ and 2900.

$\sin 20.2° = \frac{2900}{\ell}$ $\sin \theta = \frac{\text{opp}}{\text{hyp}}$

$\ell = \frac{2900}{\sin 20.2°}$ Solve for ℓ.

$\ell \approx 8398.5$ Use a calculator.

The length of the run is about 8399 feet.

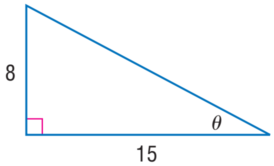
Check for Understanding

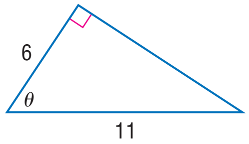
Concept Check

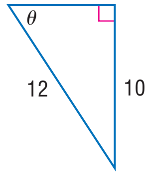
1. **Define** the word *trigonometry*.
2. **OPEN ENDED** Draw a right triangle. Label one of its acute angles θ. Then, label the hypotenuse, the leg adjacent to θ, and the leg opposite θ.
3. **Find a counterexample** to the following statement.
It is always possible to solve a right triangle.

Guided Practice

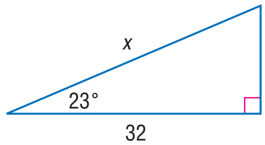
Find the values of the six trigonometric functions for angle θ.

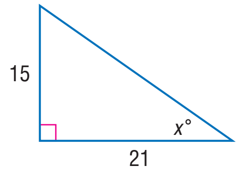
4. 

5. 

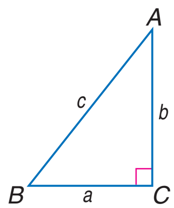
6. 

Write an equation involving sin, cos, or tan that can be used to find x. Then solve the equation. Round measures of sides to the nearest tenth and angles to the nearest degree.

7. 

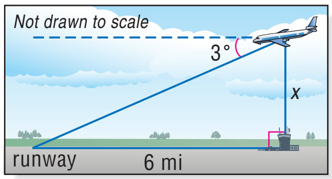
8. 

Solve $\triangle ABC$ by using the given measurements. Round measures of sides to the nearest tenth and measures of angles to the nearest degree.



9. $A = 45°, b = 6$
10. $B = 56°, c = 16$
11. $b = 7, c = 18$
12. $a = 14, b = 13$

13. **AVIATION** When landing, a jet will average a 3° angle of descent. What is the altitude x, to the nearest foot, of a jet on final descent as it passes over an airport beacon 6 miles from the start of the runway?



Standardized Test Practice

14. If $\tan \theta = 3$, find the value of $\sin \theta$.

Ⓐ $\frac{3}{10}$ Ⓑ $\frac{3\sqrt{10}}{10}$ Ⓒ $\frac{10}{3}$ Ⓓ $\frac{1}{3}$

Practice and Apply

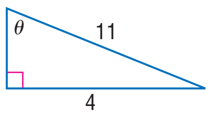
Homework Help

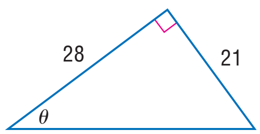
For Exercises	See Examples
15–20	1
21–24, 27, 28	3
25, 26, 37–40	5
29–36	4
41–46	6, 7
49	2

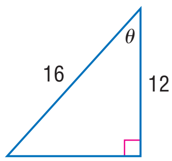
Extra Practice

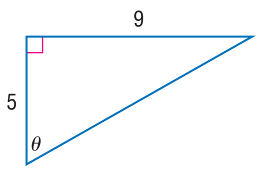
See page 857.

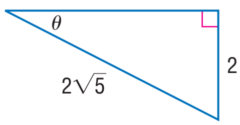
Find the values of the six trigonometric functions for angle θ.

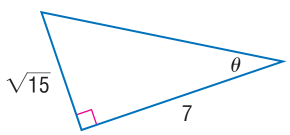
15. 

16. 

17. 

18. 

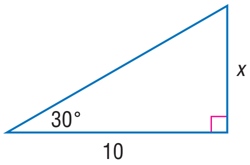
19. 

20. 

CONTENTS

Write an equation involving sin, cos, or tan that can be used to find x. Then solve the equation. Round measures of sides to the nearest tenth and measures of angles to the nearest degree.

21.



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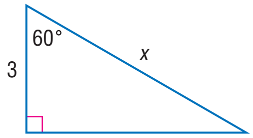
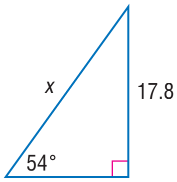
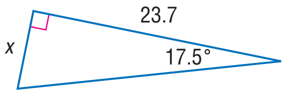


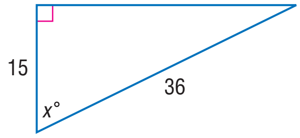
23.



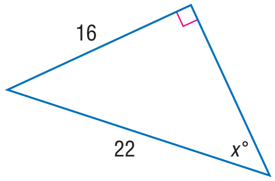
24.



25.



26.



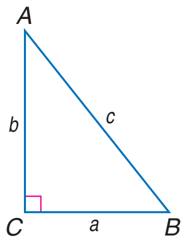
27. Using the 30°-60°-90° triangle shown on page 703, verify each value.

a. $\sin 30° = \frac{1}{2}$ **b.** $\cos 30° = \frac{\sqrt{3}}{2}$ **c.** $\sin 60° = \frac{\sqrt{3}}{2}$

28. Using the 45°-45°-90° triangle shown on page 703, verify each value.

a. $\sin 45° = \frac{\sqrt{2}}{2}$ **b.** $\cos 45° = \frac{\sqrt{2}}{2}$ **c.** $\tan 45° = 1$

Solve $\triangle ABC$ by using the given measurements. Round measures of sides to the nearest tenth and measures of angles to the nearest degree.



29. $A = 16°, c = 14$ **30.** $B = 27°, b = 7$

31. $A = 34°, a = 10$ **32.** $B = 15°, c = 25$

33. $B = 30°, b = 11$ **34.** $A = 45°, c = 7\sqrt{2}$

35. $B = 18°, a = \sqrt{15}$ **36.** $A = 10°, b = 15$

37. $b = 6, c = 13$ **38.** $a = 4, c = 9$

39. $\tan B = \frac{7}{8}, b = 7$ **40.** $\sin A = \frac{1}{3}, a = 5$

41. TRAVEL In a sightseeing boat near the base of the Horseshoe Falls at Niagara Falls, a passenger estimates the angle of elevation to the top of the falls to be 30°. If the Horseshoe Falls are 173 feet high, what is the distance from the boat to the base of the falls?

42. SURVEYING A surveyor stands 100 feet from a building and sights the top of the building at a 55° angle of elevation. Find the height of the building.

Career Choices

Surveyor

Land surveyors manage survey parties that measure distances, directions, and angles between points, lines, and contours on Earth's surface.

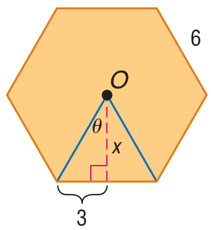
Online Research
For information about a career as a surveyor, visit: www.algebra2.com/careers

EXERCISE **For Exercises 43 and 44, use the following information.**
A preprogrammed workout on a treadmill consists of intervals walking at various rates and angles of incline. A 1% incline means 1 unit of vertical rise for every 100 units of horizontal run.

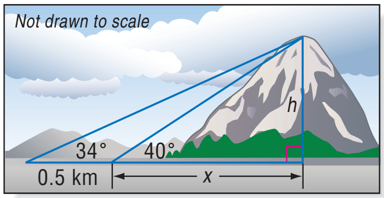
43. At what angle, with respect to the horizontal, is the treadmill bed when set at a 10% incline? Round to the nearest degree.

44. If the treadmill bed is 40 inches long, what is the vertical rise when set at an 8% incline?

45. GEOMETRY Find the area of the regular hexagon with point O as its center. (*Hint*: First find the value of x.)



46. **GEOLOGY** A geologist measured a 40° angle of elevation to the top of a mountain. After moving 0.5 kilometer farther away, the angle of elevation was 34°. How high is the top of the mountain? (*Hint*: Write a system of equations in two variables.)



WebQuest

You can use the tangent ratio to determine the maximum height of a rocket. Visit www.algebra2.com/webquest to continue work on your WebQuest project.

47. **CRITICAL THINKING** Explain why the sine and cosine of an acute angle are never greater than 1, but the tangent of an acute angle may be greater than 1.

48. **WRITING IN MATH** Answer the question that was posed at the beginning of the lesson.

How is trigonometry used in building construction?

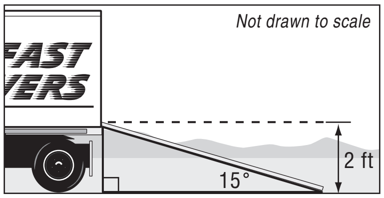
Include the following in your answer:
- an explanation as to why the ratio of vertical rise to horizontal run on an entrance ramp is the tangent of the angle the ramp makes with the horizontal, and
- an explanation of how an architect can use the tangent ratio to ensure that all the ramps he or she designs meet the ADA requirement.

Standardized Test Practice

49. If the secant of an angle θ is $\frac{25}{7}$, what is the sine of angle θ?

(A) $\frac{5}{25}$ (B) $\frac{7}{25}$ (C) $\frac{24}{25}$ (D) $\frac{25}{7}$

50. **GRID IN** The tailgate of a moving truck is 2 feet above the ground. The incline of the ramp used for loading the truck is 15° as shown. Find the length of the ramp to the nearest tenth of a foot.



Maintain Your Skills

Mixed Review

Determine whether each situation would produce a random sample. Write *yes* or *no* and explain your answer. *(Lesson 12-9)*

51. surveying band members to find the most popular type of music at your school
52. surveying people coming into a post office to find out what color cars are most popular

Find each probability if a coin is tossed 4 times. *(Lesson 12-8)*

53. P(exactly 2 heads) 54. P(4 heads) 55. P(at least 1 heads)

Solve each equation. *(Lesson 7-3)*

56. $y^4 - 64 = 0$ 57. $x^5 - 5x^3 + 4x = 0$ 58. $d + \sqrt{d} - 132 = 0$

Getting Ready for the Next Lesson

PREREQUISITE SKILL **Find each product. Include the appropriate units with your answer.** *(To review **dimensional analysis**, see Lesson 5-1.)*

59. $5 \text{ gallons}\left(\frac{4 \text{ quarts}}{1 \text{ gallon}}\right)$ 60. $6.8 \text{ miles}\left(\frac{5280 \text{ feet}}{1 \text{ mile}}\right)$

61. $\left(\frac{2 \text{ square meters}}{5 \text{ dollars}}\right) 30 \text{ dollars}$ 62. $\left(\frac{4 \text{ liters}}{5 \text{ minutes}}\right) 60 \text{ minutes}$

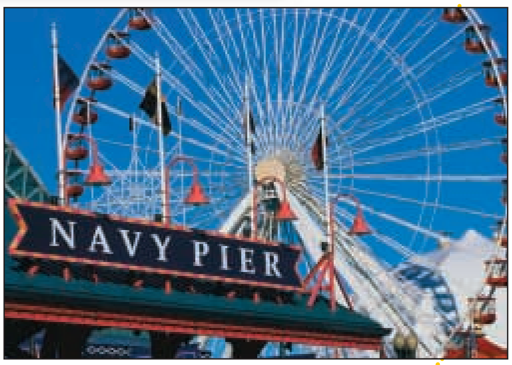
13-2 Angles and Angle Measure

What You'll Learn

- Change radian measure to degree measure and vice versa.
- Identify coterminal angles.

How can angles be used to describe circular motion?

The Ferris wheel at Navy Pier in Chicago has a 140-foot diameter and 40 gondolas equally spaced around its circumference. The average angular velocity ω of one of the gondolas is given by $\omega = \frac{\theta}{t}$, where θ is the angle through which the gondola has revolved after a specified amount of time t. For example, if a gondola revolves through an angle of 225° in 40 seconds, then its average angular velocity is 225° ÷ 40 or about 5.6° per second.

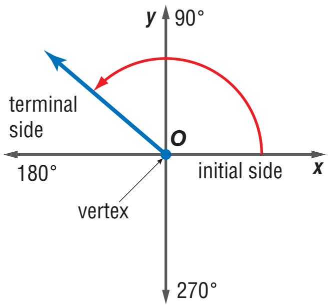


Vocabulary

- initial side
- terminal side
- standard position
- unit circle
- radian
- coterminal angles

ANGLE MEASUREMENT What does an angle measuring 225° look like? In Lesson 13-1, you worked only with acute angles, those measuring between 0° and 90°, but angles can have *any* real number measurement.

On a coordinate plane, an angle may be generated by the rotation of two rays that share a fixed endpoint at the origin. One ray, called the **initial side** of the angle, is fixed along the positive x-axis. The other ray, called the **terminal side** of the angle, can rotate about the center. An angle positioned so that its vertex is at the origin and its initial side is along the positive x-axis is said to be in **standard position**.



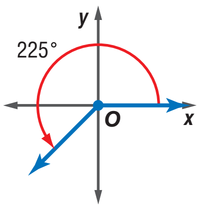
Study Tip

Reading Math

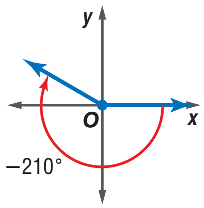
In trigonometry, an angle is sometimes referred to as an *angle of rotation*.

The measure of an angle is determined by the amount and direction of rotation from the initial side to the terminal side.

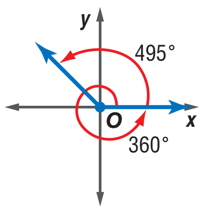
Positive Angle Measure
counterclockwise



Negative Angle Measure
clockwise



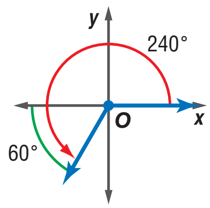
When terminal sides rotate, they may sometimes make one or more revolutions. An angle whose terminal side has made exactly one revolution has a measure of 360°.



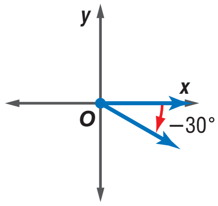
Example 1 Draw an Angle in Standard Position

Draw an angle with the given measure in standard position.

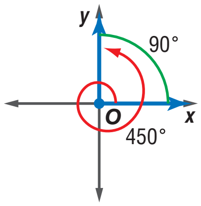
a. 240° $240° = 180° + 60°$
Draw the terminal side of the angle 60° counterclockwise past the negative x-axis.



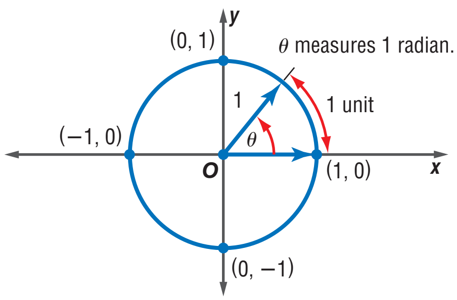
b. −30° The angle is negative.
Draw the terminal side of the angle 30° clockwise from the positive x-axis.



c. 450° $450° = 360° + 90°$
Draw the terminal side of the angle 90° counterclockwise past the positive x-axis.



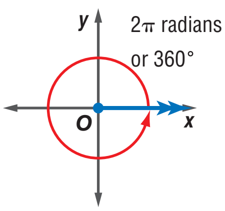
Another unit used to measure angles is a radian. The definition of a radian is based on the concept of a **unit circle**, which is a circle of radius 1 unit whose center is at the origin of a coordinate system. One **radian** is the measure of an angle θ in standard position whose rays intercept an arc of length 1 unit on the unit circle.



Study Tip

Radian Measure

As with degrees, the measure of an angle in radians is positive if its rotation is counterclockwise. The measure is negative if the rotation is clockwise.



The circumference of any circle is $2\pi r$, where r is the radius measure. So the circumference of a unit circle is $2\pi(1)$ or 2π units. Therefore, an angle representing one complete revolution of the circle measures 2π radians. This same angle measures 360°. Therefore, the following equation is true.

$$2\pi \text{ radians} = 360°$$

To change angle measures from radians to degrees or vice versa, solve the equation above in terms of both units.

$2\pi \text{ radians} = 360°$	$2\pi \text{ radians} = 360°$
$\dfrac{2\pi \text{ radians}}{2\pi} = \dfrac{360°}{2\pi}$	$\dfrac{2\pi \text{ radians}}{360} = \dfrac{360°}{360}$
$1 \text{ radian} = \dfrac{180°}{\pi}$	$\dfrac{\pi \text{ radians}}{180} = 1°$
1 radian is about 57 degrees.	1 degree is about 0.0175 radian.

These equations suggest a method for converting between radian and degree measure.

Key Concept — Radian and Degree Measure

- To rewrite the radian measure of an angle in degrees, multiply the number of radians by $\frac{180°}{\pi \text{ radians}}$.
- To rewrite the degree measure of an angle in radians, multiply the number of degrees by $\frac{\pi \text{ radians}}{180°}$.

Example 2 Convert Between Degree and Radian Measure

Rewrite the degree measure in radians and the radian measure in degrees.

a. $60°$

$$60° = 60\cancel{°}\left(\frac{\pi \text{ radians}}{180\cancel{°}}\right)$$

$$= \frac{60\pi}{180} \text{ radians or } \frac{\pi}{3}$$

b. $-\frac{7\pi}{4}$

$$-\frac{7\pi}{4} = \left(-\frac{7\cancel{\pi}}{4}\ \cancel{\text{radians}}\right)\left(\frac{180°}{\cancel{\pi}\ \cancel{\text{radians}}}\right)$$

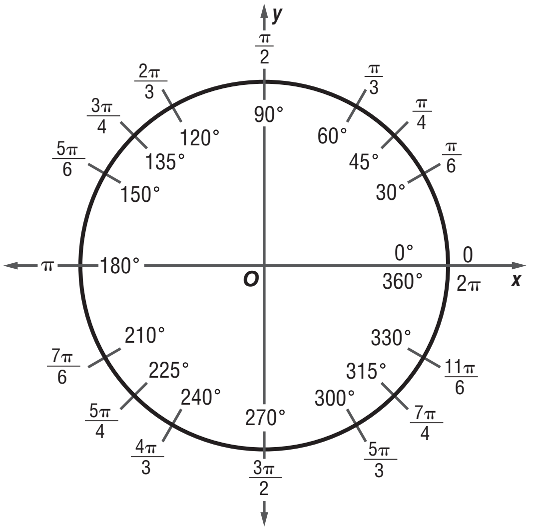
$$= -\frac{1260°}{4} \text{ or } -315°$$

Study Tip

Reading Math

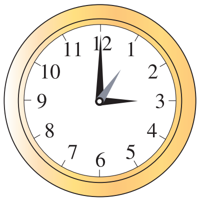
The word *radian* is usually omitted when angles are expressed in radian measure. Thus, when no units are given for an angle measure, radian measure is implied.

You will find it useful to learn equivalent degree and radian measures for the special angles shown in the diagram at the right. This diagram is more easily learned by memorizing the equivalent degree and radian measures for the first quadrant and for 90°. All of the other special angles are multiples of these angles.



Example 3 Measure an Angle in Degrees and Radians

TIME **Find both the degree and radian measures of the angle through which the hour hand on a clock rotates from 1:00 P.M. to 3:00 P.M.**

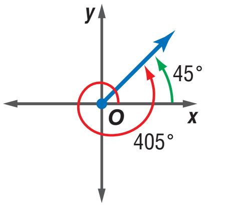


The numbers on a clock divide it into 12 equal parts with 12 equal angles. The angle from 1 to 3 on the clock represents $\frac{2}{12}$ or $\frac{1}{6}$ of a complete rotation of 360°. $\frac{1}{6}$ of 360° is 60°.

Since the rotation is clockwise, the angle through which the hour hand rotates is negative. Therefore, the angle measures $-60°$.

60° has an equivalent radian measure of $\frac{\pi}{3}$. So the equivalent radian measure of $-60°$ is $-\frac{\pi}{3}$.

COTERMINAL ANGLES If you graph a 405° angle and a 45° angle in standard position on the same coordinate plane, you will notice that the terminal side of the 405° angle is the same as the terminal side of the 45° angle. When two angles in standard position have the same terminal sides, they are called **coterminal angles**.



Notice that $405° - 45° = 360°$. In degree measure, coterminal angles differ by an integral multiple of 360°. You can find an angle that is coterminal to a given angle by adding or subtracting a multiple of 360°. In radian measure, a coterminal angle is found by adding or subtracting a multiple of 2π.

Example 4 Find Coterminal Angles

Find one angle with positive measure and one angle with negative measure coterminal with each angle.

a. 240°

A positive angle is $240° + 360°$ or $600°$.

A negative angle is $240° - 360°$ or $-120°$.

Study Tip

Coterminal Angles

Notice in Example 4b that it is necessary to subtract a multiple of 2π to find a coterminal angle with negative measure.

b. $\frac{9\pi}{4}$

A positive angle is $\frac{9\pi}{4} + 2\pi$ or $\frac{17\pi}{4}$. $\quad \frac{9\pi}{4} + \frac{8\pi}{4} = \frac{17\pi}{4}$

A negative angle is $\frac{9\pi}{4} - 2(2\pi)$ or $-\frac{7\pi}{4}$. $\quad \frac{9\pi}{4} + \left(-\frac{16\pi}{4}\right) = -\frac{7\pi}{4}$

Check for Understanding

Concept Check

1. **Name** the set of numbers to which angle measures belong.
2. **Define** the term radian.
3. **OPEN ENDED** Draw and label an example of an angle with negative measure in standard position. Then find an angle with positive measure that is coterminal with this angle.

Guided Practice

Draw an angle with the given measure in standard position.

4. 70° 5. 300° 6. 570° 7. −45°

Rewrite each degree measure in radians and each radian measure in degrees.

8. 130° 9. −10° 10. 485°

11. $\frac{3\pi}{4}$ 12. $-\frac{\pi}{6}$ 13. $\frac{19\pi}{3}$

Find one angle with positive measure and one angle with negative measure coterminal with each angle.

14. 60° 15. 425° 16. $\frac{\pi}{3}$

Application

ASTRONOMY For Exercises 17 and 18, use the following information.
Earth rotates on its axis once every 24 hours.

17. How long does it take Earth to rotate through an angle of 315°?
18. How long does it take Earth to rotate through an angle of $\frac{\pi}{6}$?

Practice and Apply

Draw an angle with the given measure in standard position.

19. 235° 20. 270° 21. 790° 22. 380°

23. −150° 24. −50° 25. π 26. $-\frac{2\pi}{3}$

Homework Help

For Exercises	See Examples
19–26	1
27–42	2
43–54	4
55–59	3

Extra Practice

See page 857.

Rewrite each degree measure in radians and each radian measure in degrees.

27. $120°$ **28.** $60°$ **29.** $-15°$ **30.** $-225°$

31. $660°$ **32.** $570°$ **33.** $158°$ **34.** $260°$

35. $\frac{5\pi}{6}$ **36.** $\frac{11\pi}{4}$ **37.** $-\frac{\pi}{4}$ **38.** $-\frac{\pi}{3}$

39. $\frac{29\pi}{4}$ **40.** $\frac{17\pi}{6}$ **41.** 9 **42.** 3

Find one angle with positive measure and one angle with negative measure coterminal with each angle.

43. $225°$ **44.** $30°$ **45.** $-15°$

46. $-140°$ **47.** $368°$ **48.** $760°$

49. $\frac{3\pi}{4}$ **50.** $\frac{7\pi}{6}$ **51.** $-\frac{5\pi}{4}$

52. $-\frac{2\pi}{3}$ **53.** $\frac{9\pi}{2}$ **54.** $\frac{17\pi}{4}$

55. DRIVING Some sport-utility vehicles (SUVs) use 15-inch radius wheels. When driven 40 miles per hour, determine the measure of the angle through which a point on the wheel travels every second. Round to both the nearest degree and nearest radian.

More About. . .

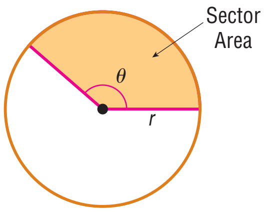
Driving

A leading U.S. automaker plans to build a hybrid sport-utility vehicle in the near future that will use an electric motor to boost fuel efficiency and reduce polluting emissions.

Source: *The Dallas Morning News*

GEOMETRY **For Exercises 56 and 57, use the following information.**

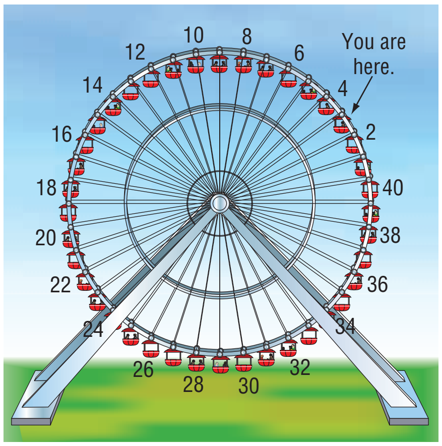
A *sector* is a region of a circle that is bounded by a central angle θ and its intercepted arc. The area A of a sector with radius r and central angle θ is given by $A = \frac{1}{2}r^2\theta$, where θ is measured in radians.



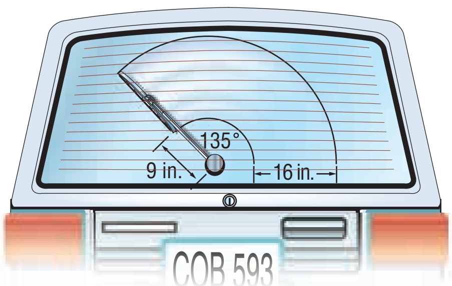
56. Find the area of a sector with a central angle of $\frac{4\pi}{3}$ radians in a circle whose radius measures 10 inches.

57. Find the area of a sector with a central angle of $150°$ in a circle whose radius measures 12 meters.

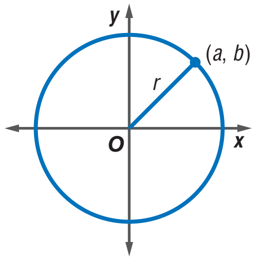
58. ENTERTAINMENT Suppose the gondolas on the Navy Pier Ferris wheel were numbered from 1 through 40 consecutively in a counterclockwise fashion. If you were sitting in gondola number 3 and the wheel were to rotate counterclockwise through $\frac{47\pi}{10}$ radians, which gondola used to be in the position that you are in now?



59. CARS Use the Area of a Sector Formula in Exercises 56 and 57 to find the area swept by the rear windshield wiper of the car shown at the right.



60. **CRITICAL THINKING** If (a, b) is on a circle that has radius r and center at the origin, prove that each of the following points is also on this circle.



a. $(a, -b)$

b. (b, a)

c. $(b, -a)$

61. **WRITING IN MATH** Answer the question that was posed at the beginning of the lesson.

How can angles be used to describe circular motion?

Include the following in your answer:

- an explanation of the significance of angles of more than 180° in terms of circular motion,
- an explanation of the significance of angles with negative measure in terms of circular motion, and
- an interpretation of a rate of more than 360° per minute.

Standardized Test Practice

62. **QUANTITATIVE COMPARISON** **Compare the quantity in Column A and the quantity in Column B. Then determine whether:**

(A) **the quantity in Column A is greater,**

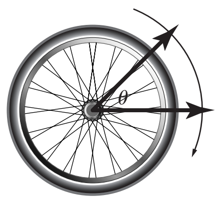
(B) **the quantity in Column B is greater,**

(C) **the two quantities are equal, or**

(D) **the relationship cannot be determined from the information given.**

Column A	Column B
56°	$\frac{14\pi}{45}$

63. Angular velocity is defined by the equation $\omega = \frac{\theta}{t}$, where θ is usually expressed in radians and t represents time. Find the angular velocity in radians per second of a point on a bicycle tire if it completes 2 revolutions in 3 seconds.

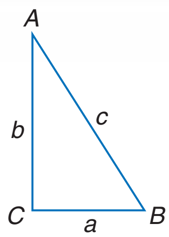


(A) $\frac{\pi}{3}$ (B) $\frac{\pi}{2}$ (C) $\frac{2\pi}{3}$ (D) $\frac{4\pi}{3}$

Maintain Your Skills

Mixed Review

Solve $\triangle ABC$ by using the given measurements. Round measures of sides to the nearest tenth and measures of angles to the nearest degree. *(Lesson 13-1)*



64. $A = 34°, b = 5$

65. $B = 68°, b = 14.7$

66. $B = 55°, c = 16$

67. $a = 0.4, b = 0.4\sqrt{3}$

Find the margin of sampling error. *(Lesson 12-9)*

68. $p = 72\%, n = 100$

69. $p = 50\%, n = 200$

Determine whether each situation involves a *permutation* or a *combination*. Then find the number of possibilities. *(Lesson 12-2)*

70. choosing an arrangement of 5 CDs from your 30 favorite CDs

71. choosing 3 different types of snack foods out of 7 at the store to take on a trip

Find $[g \circ h](x)$ and $[h \circ g](x)$. *(Lesson 7-7)*

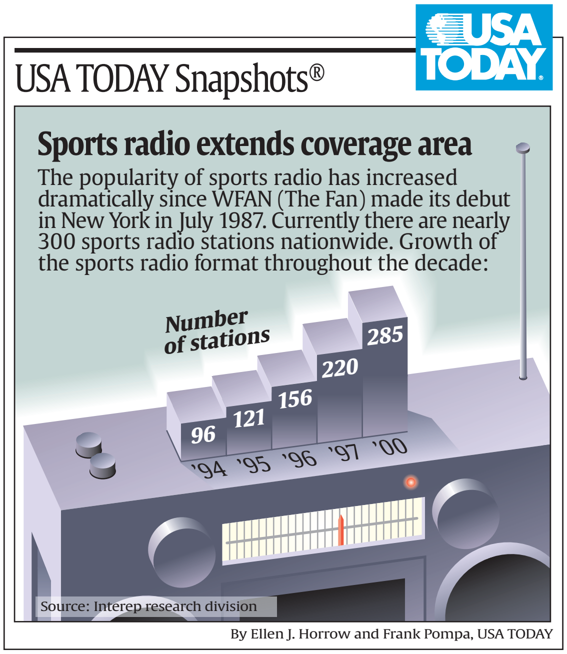
72. $g(x) = 2x$
$h(x) = 3x - 4$

73. $g(x) = 2x + 5$
$h(x) = 2x^2 - 3x + 9$

For Exercises 74 and 75, use the graph at the right.
The number of sports radio stations can be modeled by $R(x) = 7.8x^2 + 16.6x + 95.8$, where x is the number of years since 1996. *(Lesson 7-5)*

74. Use synthetic substitution to estimate the number of sports radio stations for 2006.

75. Evaluate $R(12)$. What does this value represent?



Getting Ready for the Next Lesson

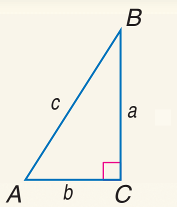
PREREQUISITE SKILL **Simplify each expression.**
*(To review **rationalizing denominators**, see Lesson 5-6.)*

76. $\dfrac{2}{\sqrt{3}}$ **77.** $\dfrac{3}{\sqrt{5}}$ **78.** $\dfrac{4}{\sqrt{6}}$

79. $\dfrac{5}{\sqrt{10}}$ **80.** $\dfrac{\sqrt{7}}{\sqrt{2}}$ **81.** $\dfrac{\sqrt{5}}{\sqrt{8}}$

Practice Quiz 1 — Lessons 13-1 and 13-2

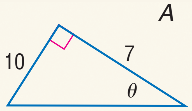
Solve $\triangle ABC$ by using the given measurements. Round measures of sides to the nearest tenth and measures of angles to the nearest degree. *(Lesson 13-1)*



1. $A = 48°, b = 12$ **2.** $a = 18, c = 21$

3. Draw an angle measuring $-60°$ in standard position. *(Lesson 13-1)*

4. Find the values of the six trigonometric functions for angle θ in the triangle at the right. *(Lesson 13-1)*



Rewrite each degree measure in radians and each radian measure in degrees. *(Lesson 13-2)*

5. $190°$ **6.** $450°$ **7.** $\dfrac{7\pi}{6}$ **8.** $-\dfrac{11\pi}{5}$

Find one angle with positive measure and one angle with negative measure conterminal with each angle. *(Lesson 13-2)*

9. $-55°$ **10.** $\dfrac{11\pi}{3}$

Investigating Regular Polygons Using Trigonometry

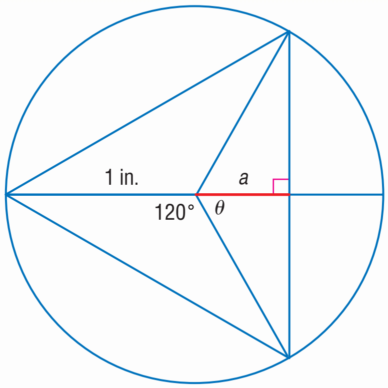
Collect the Data

- Use a compass to draw a circle with a radius of one inch. Inscribe an equilateral triangle inside of the circle. To do this, use a protractor to measure three angles of 120° at the center of the circle, since $\frac{360°}{3} = 120°$. Then connect the points where the sides of the angles intersect the circle using a straightedge.
- The **apothem** of a regular polygon is a segment that is drawn from the center of the polygon perpendicular to a side of the polygon. Use the cosine of angle θ to find the length of an apothem, labeled a in the diagram below.

Analyze the Data

1. Make a table like the one shown below and record the length of the apothem of the equilateral triangle.

Number of Sides, n	θ	a
3	60	
4	45	
5		
6		
7		
8		
9		
10		



Inscribe each regular polygon named in the table in a circle of radius one inch. Copy and complete the table.

2. What do you notice about the measure of θ as the number of sides of the inscribed polygon increases?

3. What do you notice about the values of a?

Make a Conjecture

4. Suppose you inscribe a 20-sided regular polygon inside a circle. Find the measure of angle θ.

5. Write a formula that gives the measure of angle θ for a polygon with n sides.

6. Write a formula that gives the length of the apothem of a regular polygon inscribed in a circle of radius one inch.

7. How would the formula you wrote in Exercise 6 change if the radius of the circle was not one inch?

13-3 Trigonometric Functions of General Angles

What You'll Learn

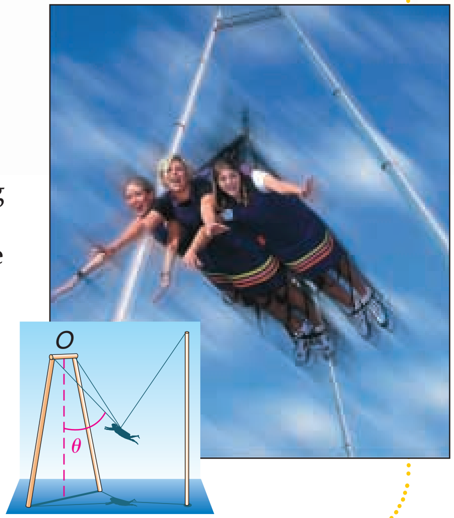
- Find values of trigonometric functions for general angles.
- Use reference angles to find values of trigonometric functions.

Vocabulary

- quadrantal angle
- reference angle

How can you model the position of riders on a skycoaster?

A skycoaster consists of a large arch from which two steel cables hang and are attached to riders suited together in a harness. A third cable, coming from a larger tower behind the arch, is attached with a ripcord. Riders are hoisted to the top of the larger tower, pull the ripcord, and then plunge toward Earth. They swing through the arch, reaching speeds of more than 60 miles per hour. After the first several swings of a certain skycoaster, the angle θ of the riders from the center of the arch is given by $\theta = 0.2 \cos(1.6t)$, where t is the time in seconds after leaving the bottom of their swing.



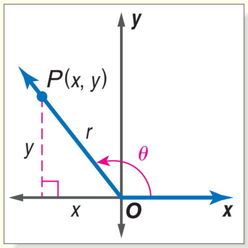
TRIGONOMETRIC FUNCTIONS AND GENERAL ANGLES In Lesson 13-1, you found values of trigonometric functions whose domains were the set of all acute angles, angles between 0 and $\frac{\pi}{2}$, of a right triangle. For $t > 0$ in the equation above, you must find the cosine of an angle greater than $\frac{\pi}{2}$. In this lesson, we will extend the domain of trigonometric functions to include angles of *any* measure.

Key Concept — *Trigonometric Functions, θ in Standard Position*

Let θ be an angle in standard position and let $P(x, y)$ be a point on the terminal side of θ. Using the Pythagorean Theorem, the distance r from the origin to P is given by $r = \sqrt{x^2 + y^2}$. The trigonometric functions of an angle in standard position may be defined as follows.

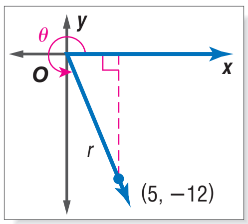
$\sin\theta = \frac{y}{r}$ $\qquad \cos\theta = \frac{x}{r}$ $\qquad \tan\theta = \frac{y}{x}, x \neq 0$

$\csc\theta = \frac{r}{y}, y \neq 0$ $\qquad \sec\theta = \frac{r}{x}, x \neq 0$ $\qquad \cot\theta = \frac{x}{y}, y \neq 0$



Example 1 *Evaluate Trigonometric Functions for a Given Point*

Find the exact values of the six trigonometric functions of θ if the terminal side of θ contains the point $(5, -12)$.



From the coordinates given, you know that $x = 5$ and $y = -12$. Use the Pythagorean Theorem to find r.

(continued on the next page)

$$r = \sqrt{x^2 + y^2} \quad \text{Pythagorean Theorem}$$
$$= \sqrt{5^2 + (-12)^2} \quad \text{Replace } x \text{ with 5 and } y \text{ with } -12.$$
$$= \sqrt{169} \text{ or } 13 \quad \text{Simplify.}$$

Now, use $x = 5$, $y = -12$, and $r = 13$ to write the ratios.

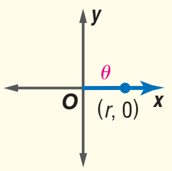
$$\sin\theta = \frac{y}{r} = \frac{-12}{13} \text{ or } -\frac{12}{13} \qquad \cos\theta = \frac{x}{r} = \frac{5}{13} \qquad \tan\theta = \frac{y}{x} = \frac{-12}{5} \text{ or } -\frac{12}{5}$$

$$\csc\theta = \frac{r}{y} = \frac{13}{-12} \text{ or } -\frac{13}{12} \qquad \sec\theta = \frac{r}{x} = \frac{13}{5} \qquad \cot\theta = \frac{x}{y} = \frac{5}{-12} \text{ or } -\frac{5}{12}$$

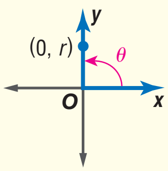
If the terminal side of angle θ lies on one of the axes, θ is called a **quadrantal angle**. The quadrantal angles are 0°, 90°, 180°, and 270°. Notice that for these angles either x or y is equal to 0. Since division by zero is undefined, two of the trigonometric values are undefined for each quadrantal angle.

Key Concept — *Quadrantal Angles*

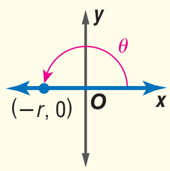
$\theta = 0°$ or 0 radians



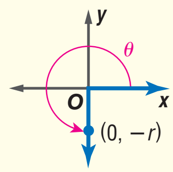
$\theta = 90°$ or $\frac{\pi}{2}$



$\theta = 180°$ or π radians

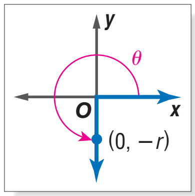


$\theta = 270°$ or $\frac{3\pi}{2}$ radians



Example 2 *Quadrantal Angles*

Find the values of the six trigonometric functions for an angle in standard position that measures 270°.



When $\theta = 270°$, $x = 0$ and $y = -r$.

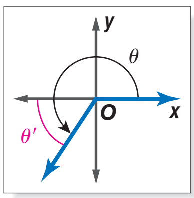
$$\sin\theta = \frac{y}{r} = \frac{-r}{r} \text{ or } -1 \qquad \cos\theta = \frac{x}{r} = \frac{0}{r} \text{ or } 0 \qquad \tan\theta = \frac{y}{x} = \frac{-r}{0} \text{ or undefined}$$

$$\csc\theta = \frac{r}{y} = \frac{r}{-r} \text{ or } -1 \qquad \sec\theta = \frac{r}{x} = \frac{r}{0} \text{ or undefined} \qquad \cot\theta = \frac{x}{y} = \frac{0}{-r} \text{ or } 0$$

Study Tip

Reading Math
θ' is read *theta prime*.

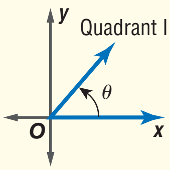
REFERENCE ANGLES To find the values of trigonometric functions of angles greater than 90° (or less than 0°), you need to know how to find the measures of reference angles. If θ is a nonquadrantal angle in standard position, its **reference angle**, θ', is defined as the acute angle formed by the terminal side of θ and the x-axis.



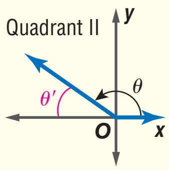
You can use the rule below to find the reference angle for any nonquadrantal angle θ where $0° < \theta < 360°$ (or $0 < \theta < 2\pi$).

Key Concept — Reference Angle Rule

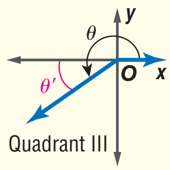
For any nonquadrantal angle θ, $0° < \theta < 360°$ (or $0 < \theta < 2\pi$), its reference angle θ' is defined as follows.



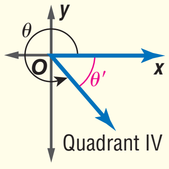
$\theta' = \theta$



$\theta' = 180° - \theta$
$(\theta' = \pi - \theta)$



$\theta' = \theta - 180°$
$(\theta' = \theta - \pi)$



$\theta' = 360° - \theta$
$(\theta' = 2\pi - \theta)$

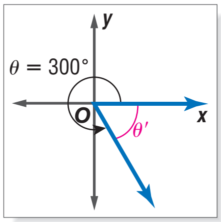
If the measure of θ is greater than 360° or less than 0°, its reference angle can be found by associating it with a coterminal angle of positive measure between 0° and 360°.

Example 3 Find the Reference Angle for a Given Angle

Sketch each angle. Then find its reference angle.

a. **300°**

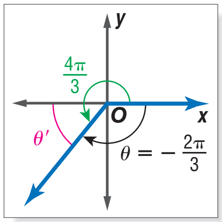
Because the terminal side of 300° lies in Quadrant IV, the reference angle is $360° - 300°$ or $60°$.



b. $-\frac{2\pi}{3}$

A coterminal angle of $-\frac{2\pi}{3}$ is $2\pi - \frac{2\pi}{3}$ or $\frac{4\pi}{3}$.

Because the terminal side of this angle lies in Quadrant III, the reference angle is $\frac{4\pi}{3} - \pi$ or $\frac{\pi}{3}$.



To use the reference angle θ' to find a trigonometric value of θ, you need to know the sign of that function for an angle θ. From the function definitions, these signs are determined by x and y, since r is always positive. Thus, the sign of each trigonometric function is determined by the quadrant in which the terminal side of θ lies.

The chart below summarizes the signs of the trigonometric functions for each quadrant.

	Quadrant			
Function	**I**	**II**	**III**	**IV**
sin θ or csc θ	+	+	–	–
cos θ or sec θ	+	–	–	+
tan θ or cot θ	+	–	+	–

Use the following steps to find the value of a trigonometric function of any angle θ.

Step 1 Find the reference angle θ'.

Step 2 Find the value of the trigonometric function for θ'.

Step 3 Using the quadrant in which the terminal side of θ lies, determine the sign of the trigonometric function value of θ.

Example 4 Use a Reference Angle to Find a Trigonometric Value

Study Tip

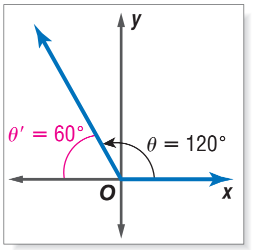
Look Back

To review **trigonometric values of angles measuring 30°, 45°, and 60°**, see Lesson 13-1.

Find the exact value of each trigonometric function.

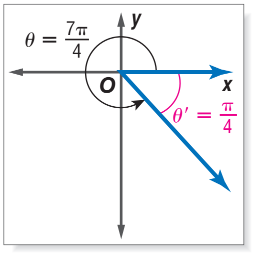
a. sin 120°

Because the terminal side of 120° lies in Quadrant II, the reference angle θ' is $180° - 120°$ or 60°. The sine function is positive in Quadrant II, so $\sin 120° = \sin 60°$ or $\frac{\sqrt{3}}{2}$.



b. $\cot \frac{7\pi}{4}$

Because the terminal side of $\frac{7\pi}{4}$ lies in Quadrant IV, the reference angle θ' is $2\pi - \frac{7\pi}{4}$ or $\frac{\pi}{4}$. The cotangent function is negative in Quadrant IV.



$$\cot \frac{7\pi}{4} = -\cot \frac{\pi}{4}$$

$$= -\cot 45° \qquad \frac{\pi}{4} \text{ radians} = 45°$$

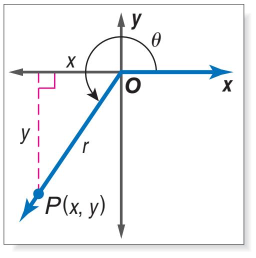
$$= -1 \qquad \cot 45° = 1$$

If you know the quadrant that contains the terminal side of θ in standard position and the exact value of one trigonometric function of θ, you can find the values of the other trigonometric functions of θ using the function definitions.

Example 5 Quadrant and One Trigonometric Value of θ

Suppose θ is an angle in standard position whose terminal side is in the Quadrant III and $\sec \theta = -\frac{4}{3}$. Find the exact values of the remaining five trigonometric functions of θ.

Draw a diagram of this angle, labeling a point $P(x, y)$ on the terminal side of θ. Use the definition of secant to find the values of x and r.



$$\sec \theta = -\frac{4}{3} \qquad \text{Given}$$

$$\frac{r}{x} = -\frac{4}{3} \qquad \text{Definition of secant}$$

Since x is negative in Quadrant III and r is always positive, $x = -3$ and $r = 4$. Use these values and the Pythagorean Theorem to find y.

$$x^2 + y^2 = r^2 \quad \text{Pythagorean Theorem}$$

$$(-3)^2 + y^2 = 4^2 \quad \text{Replace } x \text{ with } -3 \text{ and } r \text{ with } 4.$$

$$y^2 = 16 - 9 \quad \text{Simplify. Then subtract 9 from each side.}$$

$$y = \pm\sqrt{7} \quad \text{Simplify. Then take the square root of each side.}$$

$$y = -\sqrt{7} \quad y \text{ is negative in Quadrant III.}$$

Use $x = -3$, $y = -\sqrt{7}$, and $r = 4$ to write the remaining trigonometric ratios.

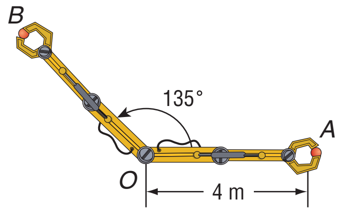
$$\sin\theta = \frac{y}{r} = \frac{-\sqrt{7}}{4} \qquad \cos\theta = \frac{x}{r} = \frac{-3}{4} \qquad \tan\theta = \frac{y}{x} = \frac{-\sqrt{7}}{-3} \text{ or } \frac{\sqrt{7}}{3}$$

$$\csc\theta = \frac{r}{y} = -\frac{4}{\sqrt{7}} \text{ or } -\frac{4\sqrt{7}}{7} \qquad \cot\theta = \frac{x}{y} = \frac{-3}{-\sqrt{7}} \text{ or } \frac{3\sqrt{7}}{7}$$

Just as an exact point on the terminal side of an angle can be used to find trigonometric function values, trigonometric function values can be used to find the exact coordinates of a point on the terminal side of an angle.

Example 6 Find Coordinates Given a Radius and an Angle

ROBOTICS In a robotics competition, a robotic arm 4 meters long is to pick up an object at point A and release it into a container at point B. The robot's owner programs the arm to rotate through an angle of precisely 135° to accomplish this task. What is the new position of the object relative to the pivot point O?



With the pivot point at the origin and the angle through which the arm rotates in standard position, point A has coordinates (0, 4). The reference angle θ' for 135° is 180° − 135° or 45°.

Let the position of point B have coordinates (x, y). Then, use the definitions of sine and cosine to find the value of x and y. The value of r is the length of the robotic arm, 4 meters. Because B is in Quadrant II, the cosine of 135° is negative.

$$\cos 135° = \frac{x}{r} \quad \text{cosine ratio}$$

$$-\cos 45° = \frac{x}{4} \quad 180° - 135° = 45°$$

$$-\frac{\sqrt{2}}{2} = \frac{x}{4} \quad \cos 45° = \frac{\sqrt{2}}{2}$$

$$-2\sqrt{2} = x \quad \text{Solve for } x.$$

$$\sin 135° = \frac{y}{r} \quad \text{sine ratio}$$

$$\sin 45° = \frac{y}{4} \quad 180° - 35° = 45°$$

$$\frac{\sqrt{2}}{2} = \frac{y}{4} \quad \sin 45° = \frac{\sqrt{2}}{2}$$

$$2\sqrt{2} = y \quad \text{Solve for } y.$$

The exact coordinates of B are $\left(-2\sqrt{2}, 2\sqrt{2}\right)$. Since $2\sqrt{2}$ is about 2.82, the object is about 2.82 meters to the left of the pivot point and about 2.82 meters in front of the pivot point.

More About. . .

Robotics

RoboCup is an annual event in which teams from all over the world compete in a series of soccer matches in various classes according to the size and intellectual capacity of their robot. The robots are programmed to react to the ball and communicate with each other.

Source: www.robocup.org

Check for Understanding

Concept Check

1. **Determine** whether the following statement is *true* or *false*. If true, explain your reasoning. If false, give a counterexample.
   The values of the secant and tangent functions for any quadrantal angle are undefined.
2. **OPEN ENDED** Give an example of an angle whose sine is negative.
3. **Explain** how the reference angle θ' is used to find the value of a trigonometric function of θ, where θ is greater than 90°.

Guided Practice

Find the exact values of the six trigonometric functions of θ if the terminal side of θ in standard position contains the given point.

4. $(-15, 8)$
5. $(-3, 0)$
6. $(4, 4)$

Sketch each angle. Then find its reference angle.

7. 235°
8. $\frac{7\pi}{4}$
9. $-240°$

Find the exact value of each trigonometric function.

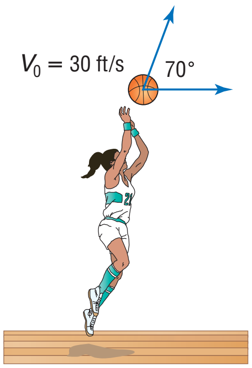
10. $\sin 300°$
11. $\cos 180°$
12. $\tan \frac{5\pi}{3}$
13. $\sec \frac{7\pi}{6}$

Suppose θ is an angle in standard position whose terminal side is in the given quadrant. For each function, find the exact values of the remaining five trigonometric functions of θ.

14. $\cos \theta = -\frac{1}{2}$, Quadrant II
15. $\cot \theta = -\frac{\sqrt{2}}{2}$, Quadrant IV

Application

16. **BASKETBALL** The maximum height H in feet that a basketball reaches after being shot is given by the formula $H = \frac{V_0^2 (\sin \theta)^2}{64}$, where V_0 represents the initial velocity in feet per second, θ represents the degree measure of the angle that the path of the basketball makes with the ground. Find the maximum height reached by a ball shot with an initial velocity of 30 feet per second at an angle of 70°.



Practice and Apply

Homework Help

For Exercises	See Examples
17–24	1
25–32	3
33–46	2, 4
47–52	5
53–55	6

Extra Practice
See page 857.

Find the exact values of the six trigonometric functions of θ if the terminal side of θ in standard position contains the given point.

17. $(7, 24)$
18. $(2, 1)$
19. $(5, -8)$
20. $(4, -3)$
21. $(0, -6)$
22. $(-1, 0)$
23. $(\sqrt{2}, -\sqrt{2})$
24. $(-\sqrt{3}, -\sqrt{6})$

Sketch each angle. Then find its reference angle.

25. 315°
26. 240°
27. $-210°$
28. $-125°$
29. $\frac{5\pi}{4}$
30. $\frac{5\pi}{6}$
31. $\frac{13\pi}{7}$
32. $-\frac{2\pi}{3}$

Find the exact value of each trigonometric function.

33. $\sin 240°$ **34.** $\sec 120°$ **35.** $\tan 300°$ **36.** $\cot 510°$

37. $\csc 5400°$ **38.** $\cos \frac{11\pi}{3}$ **39.** $\cot\left(-\frac{5\pi}{6}\right)$ **40.** $\sin \frac{3\pi}{4}$

41. $\sec \frac{3\pi}{2}$ **42.** $\csc \frac{17\pi}{6}$ **43.** $\cos(-30°)$ **44.** $\tan\left(-\frac{5\pi}{4}\right)$

45. SKYCOASTING Mikhail and Anya visit a local amusement park to ride a skycoaster. After the first several swings, the angle the skycoaster makes with the vertical is modeled by $\theta = 0.2 \cos \pi t$, with θ measured in radians and t measured in seconds. Determine the measure of the angle for $t = 0, 0.5, 1, 1.5, 2, 2.5$, and 3 in both radians and degrees.

46. NAVIGATION Ships and airplanes measure distance in nautical miles. The formula 1 nautical mile $= 6077 - 31 \cos 2\theta$ feet, where θ is the latitude in degrees, can be used to find the approximate length of a nautical mile at a certain latitude. Find the length of a nautical mile where the latitude is 60°.

More About. . .

Baseball

If a major league pitcher throws a pitch at 95 miles per hour, it takes only about 4 tenths of a second for the ball to travel the 60 feet, 6 inches from the pitcher's mound to home plate. In that time, the hitter must decide whether to swing at the ball and if so, when to swing.

Source: www.exploratorium.edu

Suppose θ is an angle in standard position whose terminal side is in the given quadrant. For each function, find the exact values of the remaining five trigonometric functions of θ.

47. $\cos \theta = \frac{3}{5}$, Quadrant IV **48.** $\tan \theta = -\frac{1}{5}$, Quadrant II

49. $\sin \theta = \frac{1}{3}$, Quadrant II **50.** $\cot \theta = \frac{1}{2}$, Quadrant III

51. $\sec \theta = -\sqrt{10}$, Quadrant III **52.** $\csc \theta = -5$, Quadrant IV

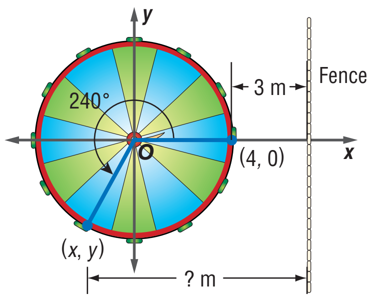
BASEBALL For Exercises 53 and 54, use the following information.

The formula $R = \frac{V_0^2 \sin 2\theta}{32}$ gives the distance of a baseball that is hit at an initial velocity of V_0 feet per second at an angle of θ with the ground.

53. If the ball was hit with an initial velocity of 80 feet per second at an angle of 30°, how far was it hit?

54. Which angle will result in the greatest distance? Explain your reasoning.

55. CAROUSELS Anthony's little brother gets on a carousel that is 8 meters in diameter. At the start of the ride, his brother is 3 meters from the fence to the ride. How far will his brother be from the fence after the carousel rotates 240°?



Online Research Data Update What is the diameter of the world's largest carousel? Visit www.algebra2.com/data_update to learn more.

CRITICAL THINKING Suppose θ is an angle in standard position with the given conditions. State the quadrant(s) in which the terminal side of θ lies.

56. $\sin \theta > 0$ **57.** $\sin \theta > 0, \cos \theta < 0$ **58.** $\tan \theta > 0, \cos \theta < 0$

59. **WRITING IN MATH** Answer the question that was posed at the beginning of the lesson.

How can you model the position of riders on a skycoaster?

Include the following in your answer:

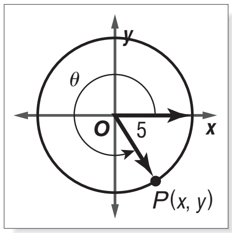
- an explanation of how you could use the cosine of the angle θ and the length of the cable from which they swing to find the horizontal position of a person on a skycoaster relative to the center of the arch, and
- an explanation of how you would use the angle θ, the height of the tower, and the length of the cable to find the height of riders from the ground.

Standardized Test Practice

60. If the cotangent of angle θ is 1, then the tangent of angle θ is

Ⓐ -1. Ⓑ 0. Ⓒ 1. Ⓓ 3.

61. **SHORT RESPONSE** Find the exact coordinates of point P, which is located at the intersection of a circle of radius 5 and the terminal side of angle θ measuring $\frac{5\pi}{3}$.



Maintain Your Skills

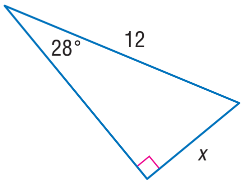
Mixed Review

Rewrite each degree measure in radians and each radian measure in degrees. *(Lesson 13-2)*

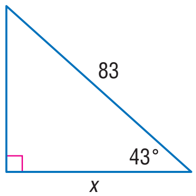
62. 90° 63. $\frac{5\pi}{3}$ 64. 5

Write an equation involving sin, cos, or tan that can be used to find x. Then solve the equation. Round measures of sides to the nearest tenth and measures of angles to the nearest degree. *(Lesson 13-1)*

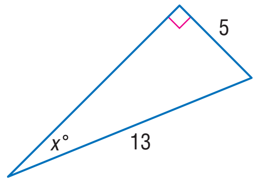
65.



66.



67.



68. **LITERATURE** In one of *Grimm's Fairy Tales*, Rumpelstiltskin has the ability to spin straw into gold. Suppose on the first day, he spun 5 pieces of straw into gold, and each day thereafter he spun twice as much. How many pieces of straw would he have spun into gold by the end of the week? *(Lesson 11-3)*

Use Cramer's Rule to solve each system of equations. *(Lesson 4-6)*

69. $3x - 4y = 13$
$-2x + 5y = -4$

70. $5x + 7y = 1$
$3x + 5y = 3$

71. $2x + 3y = -2$
$-6x + y = -34$

Getting Ready for the Next Lesson

PREREQUISITE SKILL **Solve each equation. Round to the nearest tenth.**
*(To review **solving equations with trigonometric functions**, see Lesson 13-1.)*

72. $\frac{a}{\sin 32°} = \frac{8}{\sin 65°}$ 73. $\frac{b}{\sin 45°} = \frac{21}{\sin 100°}$ 74. $\frac{c}{\sin 60°} = \frac{3}{\sin 75°}$

75. $\frac{\sin A}{14} = \frac{\sin 104°}{25}$ 76. $\frac{\sin B}{3} = \frac{\sin 55°}{7}$ 77. $\frac{\sin C}{10} = \frac{\sin 35°}{9}$

13-4 Law of Sines

What You'll Learn

- Solve problems by using the Law of Sines.
- Determine whether a triangle has one, two, or no solutions.

Vocabulary

- Law of Sines

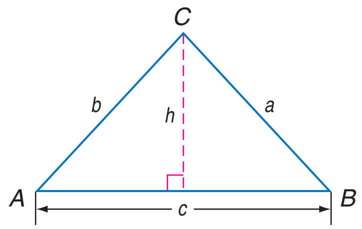
How can trigonometry be used to find the area of a triangle?

You know how to find the area of a triangle when the base and the height are known. Using this formula, the area of $\triangle ABC$ below is $\frac{1}{2}ch$. If the height h of this triangle were not known, you could still find the area given the measures of angle A and the length of side b.

$$\sin A = \frac{h}{b} \rightarrow h = b \sin A$$

By combining this equation with the area formula, you can find a new formula for the area of the triangle.

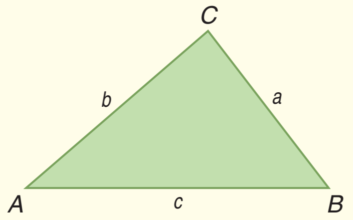
$$\text{Area} = \frac{1}{2}ch \rightarrow \text{Area} = \frac{1}{2}c(b \sin A)$$



LAW OF SINES You can find two other formulas for the area of the triangle above in a similar way. These formulas, summarized below, allow you to find the area of any triangle when you know the measures of two sides and the included angle.

Key Concept — *Area of a Triangle*

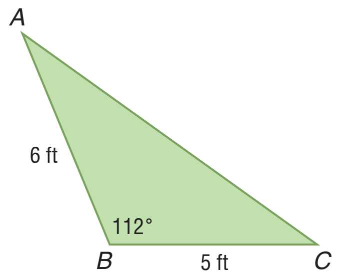
- **Words** The area of a triangle is one half the product of the lengths of two sides and the sine of their included angle.
- **Symbols**
  area $= \frac{1}{2}bc \sin A$
  area $= \frac{1}{2}ac \sin B$
  area $= \frac{1}{2}ab \sin C$



Example 1 *Find the Area of a Triangle*

Find the area of $\triangle ABC$ to the nearest tenth.

In this triangle, $a = 5$, $c = 6$, and $B = 112°$. Choose the second formula because you know the values of its variables.



$\text{Area} = \frac{1}{2}ac \sin B$	Area formula
$= \frac{1}{2}(5)(6) \sin 112°$	Replace a with 5, c with 6, and B with 112°.
≈ 13.9	Use a calculator.

To the nearest tenth, the area is 13.9 square feet.

All of the area formulas for $\triangle ABC$ represent the area of the same triangle. So, $\frac{1}{2}bc \sin A$, $\frac{1}{2}ac \sin B$, and $\frac{1}{2}ab \sin C$ are all equal. You can use this fact to derive the **Law of Sines**.

$$\frac{1}{2}bc \sin A = \frac{1}{2}ac \sin B = \frac{1}{2}ab \sin C \quad \text{Set area formulas equal to each other.}$$

$$\frac{\frac{1}{2}bc \sin A}{\frac{1}{2}abc} = \frac{\frac{1}{2}ac \sin B}{\frac{1}{2}abc} = \frac{\frac{1}{2}ab \sin C}{\frac{1}{2}abc} \quad \text{Divide each expression by } \tfrac{1}{2}abc.$$

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \quad \text{Simplify.}$$

Study Tip

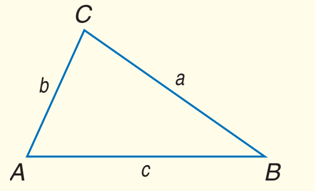
Alternate Representations

The Law of Sines may also be written as $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$.

Key Concept — *Law of Sines*

Let $\triangle ABC$ be any triangle with a, b, and c representing the measures of sides opposite angles with measurements A, B, and C respectively. Then,

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$



The Law of Sines can be used to write three different equations.

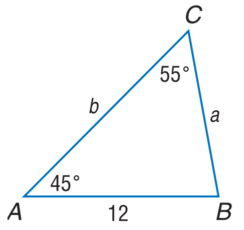
$$\frac{\sin A}{a} = \frac{\sin B}{b} \quad \text{or} \quad \frac{\sin B}{b} = \frac{\sin C}{c} \quad \text{or} \quad \frac{\sin A}{a} = \frac{\sin C}{c}$$

In Lesson 13-1, you learned how to solve right triangles. To solve *any* triangle, you can apply the Law of Sines if you know

- the measures of two angles and any side or
- the measures of two sides and the angle opposite one of them.

Example 2 Solve a Triangle Given Two Angles and a Side

Solve $\triangle ABC$.



You are given the measures of two angles and a side. First, find the measure of the third angle.

$45° + 55° + B = 180°$ — The sum of the angle measures of a triangle is 180°.

$B = 80°$ — $180 - (45 + 55) = 80$

Now use the Law of Sines to find a and b. Write two equations, each with one variable.

$\frac{\sin A}{a} = \frac{\sin C}{c}$	Law of Sines	$\frac{\sin B}{b} = \frac{\sin C}{c}$
$\frac{\sin 45°}{a} = \frac{\sin 55°}{12}$	Replace A with 45°, B with 80°, C with 55°, and c with 12.	$\frac{\sin 80°}{b} = \frac{\sin 55°}{12}$
$a = \frac{12 \sin 45°}{\sin 55°}$	Solve for the variable.	$b = \frac{12 \sin 80°}{\sin 55°}$
$a \approx 10.4$	Use a calculator.	$b \approx 14.4$

Therefore, $B = 80°$, $a \approx 10.4$, and $b \approx 14.4$.

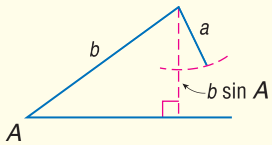
ONE, TWO, OR NO SOLUTIONS When solving a triangle, you must analyze the data you are given to determine whether there is a solution. For example, if you are given the measures of two angles and a side, as in Example 2, the triangle has a unique solution. However, if you are given the measures of two sides and the angle opposite one of them, a single solution may not exist. One of the following will be true.

- No triangle exists, and there is no solution.
- Exactly one triangle exists, and there is one solution.
- Two triangles exist, and there are two solutions.

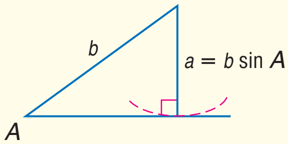
Key Concept — Possible Triangles Given Two Sides and One Opposite Angle

Suppose you are given a, b, and A for a triangle.

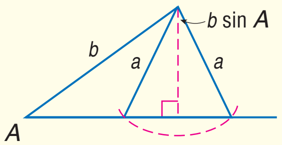
A Is Acute ($A < 90°$).



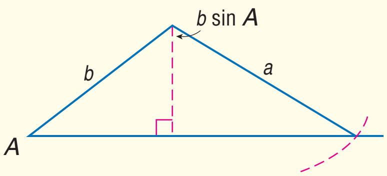
$a < b \sin A$
no solution



$a = b \sin A$
one solution

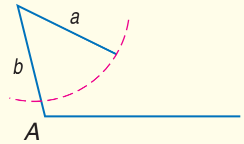


$b > a > b \sin A$
two solutions

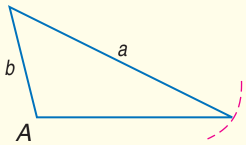


$a \geq b$
one solution

A Is Right or Obtuse ($A \geq 90°$).



$a \leq b$
no solution



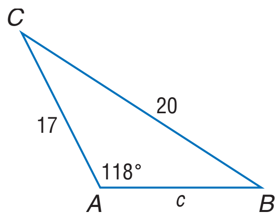
$a > b$
one solution

Example 3 One Solution

In $\triangle ABC$, $A = 118°$, $a = 20$, and $b = 17$. Determine whether $\triangle ABC$ has *no* solution, *one* solution, or *two* solutions. Then solve $\triangle ABC$.

Because angle A is obtuse and $a > b$, you know that one solution exists.

Make a sketch and then use the Law of Sines to find B.



$\frac{\sin B}{17} = \frac{\sin 118°}{20}$ Law of Sines

$\sin B = \frac{17 \sin 118°}{20}$ Multiply each side by 17.

$\sin B \approx 0.7505$ Use a calculator.

$B \approx 49°$ Use the $\sin^{-1}$ function.

The measure of angle C is approximately $180 - (118 + 49)$ or $13°$.

Use the Law of Sines again to find c.

$\frac{\sin 13}{c} = \frac{\sin 118°}{20}$ Law of Sines

$c = \frac{20 \sin 13°}{\sin 118°}$ or about 5.1 Use a calculator.

Therefore, $B \approx 49°$, $C \approx 13°$, and $c \approx 5.1$.

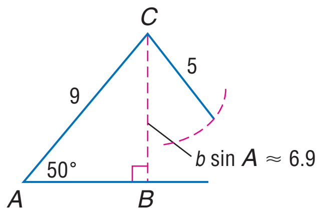
www.algebra2.com/extra_examples

CONTENTS

Example 4 No Solution

In $\triangle ABC$, $A = 50°$, $a = 5$, and $b = 9$. Determine whether $\triangle ABC$ has *no* solution, *one* solution, or *two* solutions. Then solve $\triangle ABC$.

Since angle A is acute, find $b \sin A$ and compare it with a.



$b \sin A = 9 \sin 50°$ Replace b with 9 and A with 50°.

≈ 6.9 Use a calculator.

Since $5 < 6.9$, there is no solution.

Study Tip

A Is Acute

We compare $b \sin A$ to a because $b \sin A$ is the minimum distance from C to $\overline{AB}$ when A is acute.

When two solutions for a triangle exist, it is called the *ambiguous case*.

Example 5 Two Solutions

In $\triangle ABC$, $A = 39°$, $a = 10$, and $b = 14$. Determine whether $\triangle ABC$ has *no* solution, *one* solution, or *two* solutions. Then solve $\triangle ABC$.

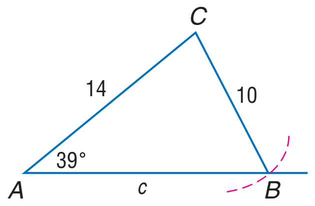
Since angle A is acute, find $b \sin A$ and compare it with a.

$b \sin A = 14 \sin 39°$ Replace b with 14 and A with 39°.

≈ 8.81 Use a calculator.

Since $14 > 10 > 8.81$, there are two solutions. Thus, there are two possible triangles to be solved.

Case 1 Acute Angle B



First, use the Law of Sines to find B.

$$\frac{\sin B}{14} = \frac{\sin 39°}{10}$$

$$\sin B = \frac{14 \sin 39°}{10}$$

$$\sin B = 0.8810$$

$$B \approx 62°$$

The measure of angle C is approximately $180 - (39 + 62)$ or $79°$.

Use the Law of Sines again to find c.

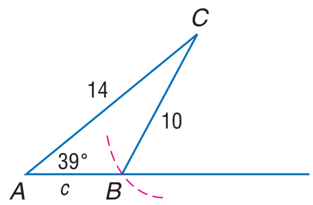
$$\frac{\sin 79°}{c} = \frac{\sin 39°}{10}$$

$$c = \frac{10 \sin 79°}{\sin 39°}$$

$$c \approx 15.6$$

Therefore, $B \approx 62°$, $C \approx 79°$, and $c \approx 15.6$.

Case 2 Obtuse Angle B



To find B, you need to find an obtuse angle whose sine is also 0.8810. To do this, subtract the angle given by your calculator, 62°, from 180°. So B is approximately $180 - 62$ or $118°$.

The measure of angle C is approximately $180 - (39 + 118)$ or $23°$.

Use the Law of Sines to find c.

$$\frac{\sin 23°}{c} = \frac{\sin 39°}{10}$$

$$c = \frac{10 \sin 23°}{\sin 39°}$$

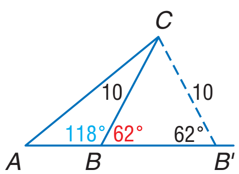
$$c \approx 6.2$$

Therefore, $B \approx 118°$, $C \approx 23°$, and $c \approx 6.2$.

Study Tip

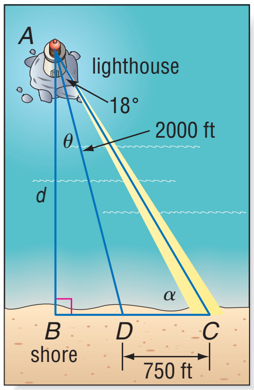
Alternate Method

Another way to find the obtuse angle in Case 2 of Example 5 is to notice in the figure below that $\triangle CBB'$ is isosceles. Since the base angles of an isosceles triangle are always congruent and $m\angle B' = 62°$, $m\angle CBB' = 62°$. Also, $\angle ABC$ and $m\angle CBB'$ are supplementary. Therefore, $m\angle ABC = 180° - 62°$ or $118°$.



Example 6 Use the Law of Sines to Solve a Problem

LIGHTHOUSES A lighthouse is located on a rock at a certain distance from a straight shore. The light revolves counterclockwise at a steady rate of one revolution per minute. As the beam revolves, it strikes a point on the shore that is 2000 feet from the lighthouse. Three seconds later, the light strikes a point 750 feet further down the shore. To the nearest foot, how far is the lighthouse from the shore?



Because the lighthouse makes one revolution every 60 seconds, the angle through which the light revolves in 3 seconds is $\frac{3}{60}(360°)$ or 18°.

Use the Law of Sines to find the measure of angle α.

$$\frac{\sin \alpha}{2000} = \frac{\sin 18°}{750} \quad \text{Law of Sines}$$

$$\sin \alpha = \frac{2000 \sin 18°}{750} \quad \text{Multiply each side by 2000.}$$

$$\sin \alpha \approx 0.8240 \quad \text{Use a calculator.}$$

$$\alpha \approx 55° \quad \text{Use the } \sin^{-1} \text{ function.}$$

Use this angle measure to find the measure of angle θ. Since $\triangle ABC$ is a right triangle, the measures of angle α and $\angle BAC$ are complementary.

$$\alpha + m\angle BAC = 90° \quad \text{Angles } \alpha \text{ and } \angle BAC \text{ are complementary.}$$

$$55° + (\theta + 18°) \approx 90° \quad \alpha \approx 55° \text{ and } m\angle BAC = \theta + 18°$$

$$\theta + 73° \approx 90° \quad \text{Simplify.}$$

$$\theta \approx 17° \quad \text{Solve for } \theta.$$

To find the distance from the lighthouse to the shore, solve $\triangle ABD$ for d.

$$\cos \theta = \frac{AB}{AD} \quad \text{Cosine ratio}$$

$$\cos 17° \approx \frac{d}{2000} \quad \theta = 17° \text{ and } AD = 2000$$

$$d \approx 2000 \cos 17° \quad \text{Solve for } d.$$

$$d \approx 1913 \quad \text{Use a calculator.}$$

The distance from the lighthouse to the shore, to the nearest foot, is 1913 feet. This answer is reasonable since 1913 is less than 2000.

More About. . .

Lighthouses

Standing 208 feet tall, the Cape Hatteras Lighthouse in North Carolina is the tallest lighthouse in the United States.

Source: www.oldcapehatteraslighthouse.com

Check for Understanding

Concept Check

1. **Determine** whether the following statement is *sometimes*, *always* or *never* true. Explain your reasoning.

   If given the measure of two sides of a triangle and the angle opposite one of them, you will be able to find a unique solution.

2. **OPEN ENDED** Give an example of a triangle that has two solutions by listing measures for A, a, and b, where a and b are in centimeters. Then draw both cases using a ruler and protractor.

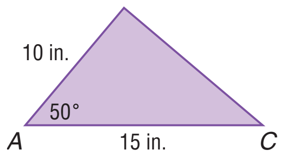
3. **FIND THE ERROR** Dulce and Gabe are finding the area of $\triangle ABC$ for $A = 64°$, $a = 15$ meters, and $b = 8$ meters using the sine function.

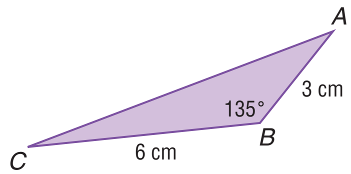
Dulce	Gabe
Area $= \frac{1}{2}(15)(8)\sin 64°$ $\approx 53.9 \text{ m}^2$	There is not enough information to find the area of ΔABC.

Who is correct? Explain your reasoning.

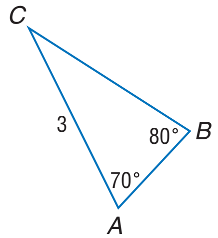
Guided Practice

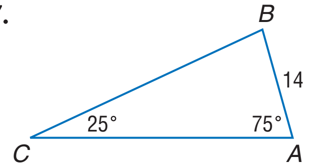
Find the area of $\triangle ABC$ to the nearest tenth.

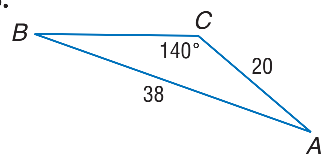
4. 

5. 

Solve each triangle. Round measures of sides to the nearest tenth and measures of angles to the nearest degree.

6. 

7. 

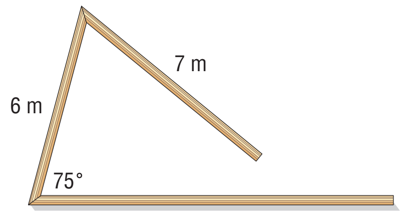
8. 

Determine whether each triangle has *no* solution, *one* solution, or *two* solutions. Then solve each triangle. Round measures of sides to the nearest tenth and measures of angles to the nearest degree.

9. $A = 123°$, $a = 12$, $b = 23$
10. $A = 30°$, $a = 3$, $b = 4$
11. $A = 55°$, $a = 10$, $b = 5$
12. $A = 145°$, $a = 18$, $b = 10$

Application

13. **WOODWORKING** Latisha is constructing a triangular brace from three beams of wood. She is to join the 6-meter beam to the 7-meter beam so that angle opposite the 7-meter beam measures 75°. To what length should Latisha cut the third beam in order to form the triangular brace? Round to the nearest tenth.



Practice and Apply

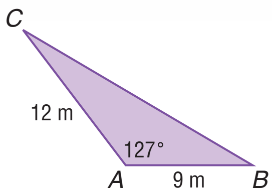
Homework Help

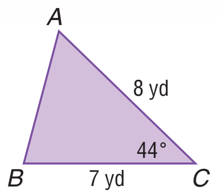
For Exercises	See Examples
14–19	1
20–37	2–5
38–41	6

Extra Practice

See page 858.

Find the area of $\triangle ABC$ to the nearest tenth.

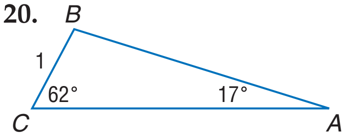
14. 

15. 

16. $B = 85°$, $c = 23$ ft, $a = 50$ ft
17. $A = 60°$, $b = 12$ cm, $c = 12$ cm
18. $C = 136°$, $a = 3$ m, $b = 4$ m
19. $B = 32°$, $a = 11$ mi, $c = 5$ mi

Solve each triangle. Round measures of sides to the nearest tenth and measures of angles to the nearest degree.

20.



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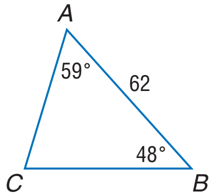

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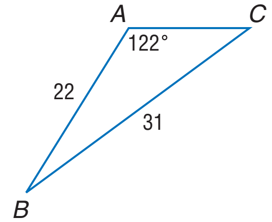

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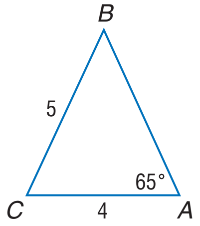

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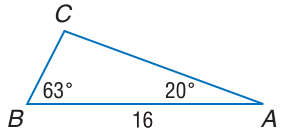

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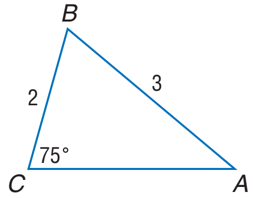


26. $A = 50°, a = 2.5, c = 3$

27. $B = 18°, C = 142°, b = 20$

Determine whether each triangle has *no* solution, *one* solution, or *two* solutions. Then solve each triangle. Round measures of sides to the nearest tenth and measures of angles to the nearest degree.

28. $A = 124°, a = 1, b = 2$

29. $A = 99°, a = 2.5, b = 1.5$

30. $A = 33°, a = 2, b = 3.5$

31. $A = 68°, a = 3, b = 5$

32. $A = 30°, a = 14, b = 28$

33. $A = 61°, a = 23, b = 8$

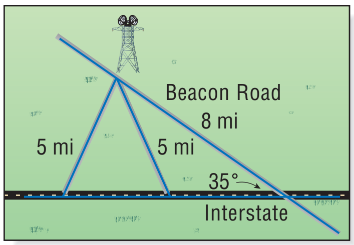
34. $A = 52°, a = 190, b = 200$

35. $A = 80°, a = 9, b = 9.1$

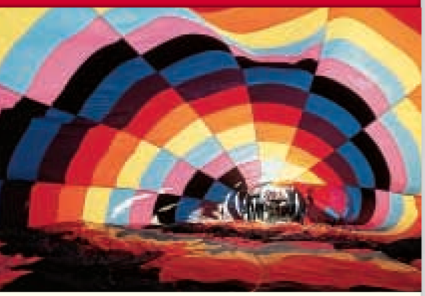
36. $A = 28°, a = 8.5, b = 7.2$

37. $A = 47°, a = 67, b = 83$

38. RADIO A radio station providing local tourist information has its transmitter on Beacon Road, 8 miles from where it intersects with the interstate highway. If the radio station has a range of 5 miles, between what two distances from the intersection can cars on the interstate tune in to hear this information?



More About. . .



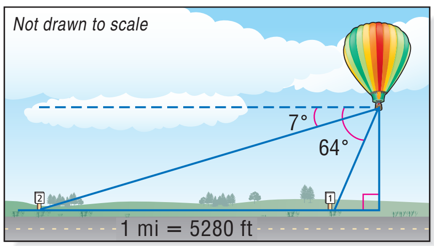
Ballooning

Hot-air balloons range in size from approximately 54,000 cubic feet to over 250,000 cubic feet.

Source: www.unicorn-ballon.com

39. FORESTRY Two forest rangers, 12 miles from each other on a straight service road, both sight an illegal bonfire away from the road. Using their radios to communicate with each other, they determine that the fire is between them. The first ranger's line of sight to the fire makes an angle of 38° with the road, and the second ranger's line of sight to the fire makes a 63° angle with the road. How far is the fire from each ranger?

40. BALLOONING As a hot-air balloon crosses over a straight portion of interstate highway, its pilot eyes two consecutive mileposts on the same side of the balloon. When viewing the mileposts the angles of depression are 64° and 7°. How high is the balloon to the nearest foot?



41. **NAVIGATION** Two fishing boats, A and B, are anchored 4500 feet apart in open water. A plane flies at a constant speed in a straight path directly over the two boats, maintaining a constant altitude. At one point during the flight, the angle of depression to A is 85°, and the angle of depression to B is 25°. Ten seconds later the plane has passed over A and spots B at a 35° angle of depression. How fast is the plane flying?

42. **CRITICAL THINKING** Given $\triangle ABC$, if $a = 20$ and $B = 47°$, then determine all possible values of b so that the triangle has

a. two solutions. **b.** one solution. **c.** no solutions.

43. **WRITING IN MATH** Answer the question that was posed at the beginning of the lesson.

How can trigonometry be used to find the area of a triangle?

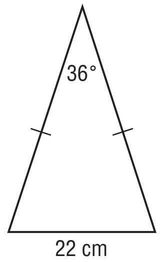
Include the following in your answer:

- the conditions that would indicate that trigonometry is needed to find the area of a triangle,
- an example of a real-world situation in which you would need trigonometry to find the area of a triangle, and
- a derivation of one of the other two area formulas.

Standardized Test Practice

44. Which of the following is the perimeter of the triangle shown?

(A) 49.0 cm (B) 66.0 cm
(C) 91.4 cm (D) 93.2 cm



45. **SHORT RESPONSE** The longest side of a triangle is 67 inches. Two angles have measures of 47° and 55°. Solve the triangle.

Maintain Your Skills

Mixed Review

Find the exact value of each trigonometric function. *(Lesson 13-3)*

46. cos 30° 47. $\cot\left(\frac{\pi}{3}\right)$ 48. $\csc\left(\frac{\pi}{4}\right)$

Find one angle with positive measure and one angle with negative measure coterminal with each angle. *(Lesson 13-2)*

49. 300° 50. 47° 51. $\frac{5\pi}{6}$

Two cards are drawn from a deck of cards. Find each probability. *(Lesson 12-5)*

52. P(both 5s or both spades) 53. P(both 7s or both red)

54. **AERONAUTICS** A rocket rises 20 feet in the first second, 60 feet in the second second, and 100 feet in the third second. If it continues at this rate, how many feet will it rise in the 20th second? *(Lesson 11-1)*

Getting Ready for the Next Lesson

PREREQUISITE SKILL **Solve each equation. Round to the nearest tenth.**
*(To review **solving equations with trigonometric functions**, see Lesson 13-1.)*

55. $a^2 = 3^2 + 5^2 - 2(3)(5) \cos 85°$ 56. $c^2 = 12^2 + 10^2 - 2(12)(10) \cos 40°$

57. $7^2 = 11^2 + 9^2 - 2(11)(9) \cos B°$ 58. $13^2 = 8^2 + 6^2 - 2(8)(6) \cos A°$

13-5 Law of Cosines

What You'll Learn

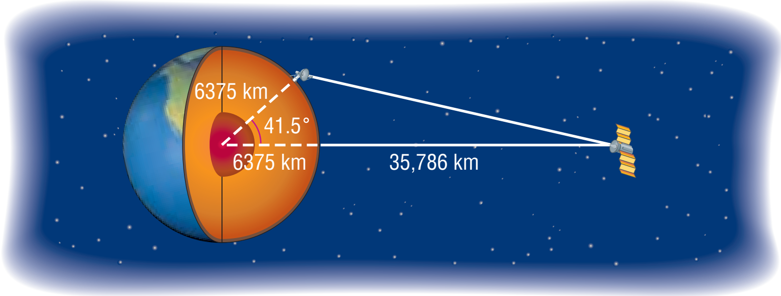
- Solve problems by using the Law of Cosines.
- Determine whether a triangle can be solved by first using the Law of Sines or the Law of Cosines.

Vocabulary

- Law of Cosines

How can you determine the angle at which to install a satellite dish?

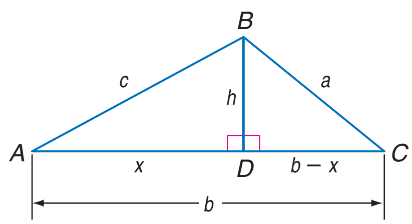
The GE-3 satellite is in a *geosynchronous orbit* about Earth, meaning that it circles Earth once each day. As a result, the satellite appears to remain stationary over one point on the equator. A receiving dish for the satellite can be directed at one spot in the sky. The satellite orbits 35,786 kilometers above the equator at 87°W longitude. The city of Valparaiso, Indiana, is located at approximately 87°W longitude and 41.5°N latitude.



Knowing the radius of Earth to be about 6375 kilometers, a satellite dish installer can use trigonometry to determine the angle at which to direct the receiver.

LAW OF COSINES Problems such as this, in which you know the measures of two sides and the included angle of a triangle, cannot be solved using the Law of Sines. You can solve problems such as this by using the **Law of Cosines**.

To derive the Law of Cosines, consider $\triangle ABC$. What relationship exists between a, b, c, and A?

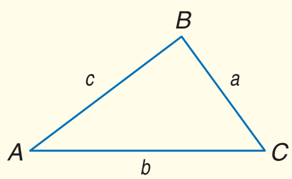


$a^2 = (b - x)^2 + h^2$	Use the Pythagorean Theorem for $\triangle DBC$.
$= b^2 - 2bx + x^2 + h^2$	Expand $(b - x)^2$.
$= b^2 - 2bx + c^2$	In $\triangle ADB$, $c^2 = x^2 + h^2$.
$= b^2 - 2b(c \cos A) + c^2$	$\cos A = \frac{x}{c}$, so $x = c \cos A$.
$= b^2 + c^2 - 2bc \cos A$	Commutative Property

Key Concept — Law of Cosines

Let $\triangle ABC$ be any triangle with a, b, and c representing the measures of sides, and opposite angles with measures A, B, and C, respectively. Then the following equations are true.

$$a^2 = b^2 + c^2 - 2bc \cos A$$
$$b^2 = a^2 + c^2 - 2ac \cos B$$
$$c^2 = a^2 + b^2 - 2ab \cos C$$



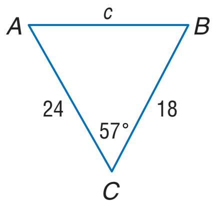
CONTENTS

You can apply the Law of Cosines to a triangle if you know

- the measures of two sides and the included angle, or
- the measures of three sides.

Example 1 Solve a Triangle Given Two Sides and Included Angle

Solve $\triangle ABC$.



You are given the measure of two sides and the included angle. Begin by using the Law of Cosines to determine c.

$c^2 = a^2 + b^2 - 2ab \cos C$ Law of Cosines

$c^2 = 18^2 + 24^2 - 2(18)(24) \cos 57°$ $a = 18$, $b = 24$, and $C = 57°$

$c^2 \approx 429.4$ Simplify using a calculator.

$c \approx 20.7$ Take the square root of each side.

Next, you can use the Law of Sines to find the measure of angle A.

$\frac{\sin A}{a} = \frac{\sin C}{c}$ Law of Sines

$\frac{\sin A}{18} \approx \frac{\sin 57°}{20.7}$ $a = 18$, $C = 57°$, and $c \approx 20.7$

$\sin A \approx \frac{18 \sin 57°}{20.7}$ Multiply each side by 18.

$\sin A \approx 0.7293$ Use a calculator.

$A \approx 47°$ Use the $\sin^{-1}$ function.

The measure of the angle B is approximately $180° - (57° + 47°)$ or $76°$. Therefore, $c \approx 20.7$, $A \approx 47°$, and $B \approx 76°$.

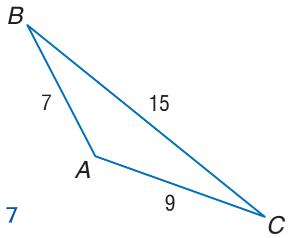
Study Tip

Alternate Method

After finding the measure of c in Example 1, the Law of Cosines could be used again to find a second angle.

Example 2 Solve a Triangle Given Three Sides

Solve $\triangle ABC$.



You are given the measures of three sides. Use the Law of Cosines to find the measure of the largest angle first, angle A.

$a^2 = b^2 + c^2 - 2bc \cos A$ Law of Cosines

$15^2 = 9^2 + 7^2 - 2(9)(7) \cos A$ $a = 15$, $b = 9$, and $c = 7$

$15^2 - 9^2 - 7^2 = -2(9)(7) \cos A$ Subtract 9^2 and 7^2 from each side.

$\frac{15^2 - 9^2 - 7^2}{-2(9)(7)} = \cos A$ Divide each side by $-2(9)(7)$.

$-0.7540 \approx \cos A$ Use a calculator.

$139° \approx A$ Use the $\cos^{-1}$ function.

You can use the Law of Sines to find the measure of angle B.

$\frac{\sin B}{b} = \frac{\sin A}{a}$ Law of Sines

$\frac{\sin B}{9} \approx \frac{\sin 139°}{15}$ $b = 9$, $A \approx 139°$, and $a = 15$

$\sin B \approx \frac{9 \sin 139°}{15}$ Multiply each side by 9.

$\sin B \approx 0.3936$ Use a calculator.

$B \approx 23°$ Use the $\sin^{-1}$ function.

The measure of the angle C is approximately $180° - (139° + 23°)$ or $18°$. Therefore, $A \approx 139°$, $B \approx 23°$, and $C \approx 18°$.

Study Tip

Sides and Angles

When solving triangles, remember that the angle with the greatest measure is always opposite the longest side. The angle with the least measure is always opposite the shortest side.

CHOOSE THE METHOD To solve a triangle that is *oblique*, or having no right angle, you need to know the measure of at least one side and any two other parts. If the triangle has a solution, then you must decide whether to begin solving by using the Law of Sines or by using the Law of Cosines. Use the chart below to help you choose.

Concept Summary — Solving an Oblique Triangle

Given	Begin by Using
two angles and any side	Law of Sines
two sides and an angle opposite one of them	Law of Sines
two sides and their included angle	Law of Cosines
three sides	Law of Cosines

More About. . .

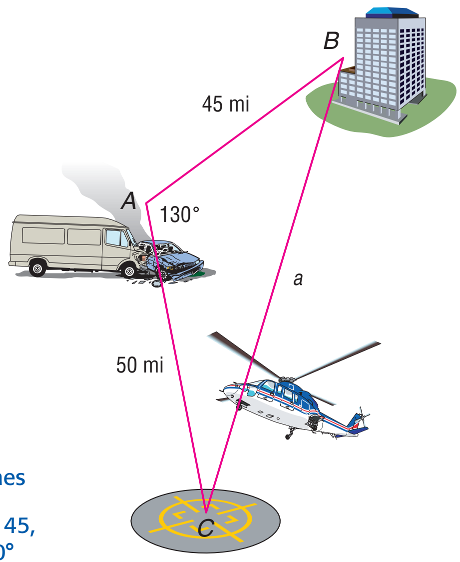
Emergency Medicine

Medical evacuation (Medevac) helicopters provide quick transportation from areas that are difficult to reach by any other means. These helicopters can cover long distances and are primary emergency vehicles in locations where there are few hospitals.

Source: The Helicopter Education Center

Example 3 Apply the Law of Cosines

EMERGENCY MEDICINE **A medical rescue helicopter has flown from its home base at point C to pick up an accident victim at point B and then from there to the hospital at point A. The pilot needs to know how far he is now from his home base so he can decide whether to refuel before returning. How far is the hospital from the helicopter's base?**



You are given the measures of two sides and their included angle, so use the Law of Cosines to find a.

$a^2 = b^2 + c^2 - 2bc \cos A$ Law of Cosines

$a^2 = 50^2 + 45^2 - 2(50)(45) \cos 130°$ $b = 50$, $c = 45$, and $A = 130°$

$a^2 \approx 7417.5$ Use a calculator to simplify.

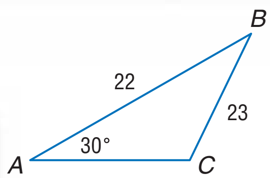
$a \approx 86.1$ Take the square root of each side.

The distance between the hospital and the helicopter base is approximately 86.1 miles.

Maintain Your Skills

Concept Check

1. **FIND THE ERROR** Mateo and Amy are deciding which method, the Law of Sines or the Law of Cosines, should be used first to solve $\triangle ABC$.



Mateo	Amy
Begin by using the Law of Sines, since you are given two sides and an angle opposite one of them.	Begin by using the Law of Cosines, since you are given two sides and their included angle.

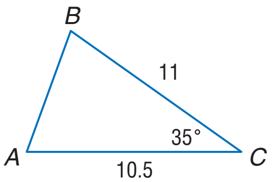
Who is correct? Explain your reasoning.

2. **Explain** how to solve a triangle by using the Law of Cosines if the lengths of
   a. three sides are known.
   b. two sides and the measure of the angle between them are known.
3. **OPEN ENDED** Give an example of a triangle that can be solved by first using the Law of Cosines.

Guided Practice

Determine whether each triangle should be solved by beginning with the Law of Sines or Law of Cosines. Then solve each triangle. Round measures of sides to the nearest tenth and measures of angles to the nearest degree.

4.



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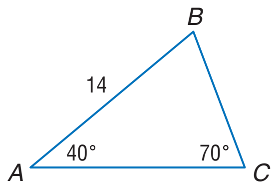
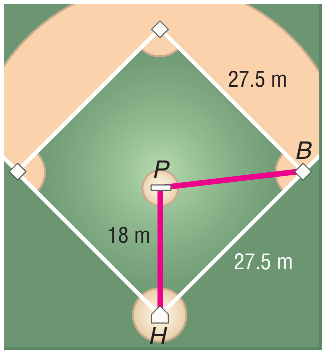


6. $A = 42°$, $b = 57$, $a = 63$

7. $a = 5$, $b = 12$, $c = 13$

Application

BASEBALL **For Exercises 8 and 9, use the following information.**
In Australian baseball, the bases lie at the vertices of a square 27.5 meters on a side and the pitcher's mound is 18 meters from home plate.



8. Find the distance from the pitcher's mound to first base.
9. Find the angle between home plate, the pitcher's mound, and first base.

Practice and Apply

Homework Help

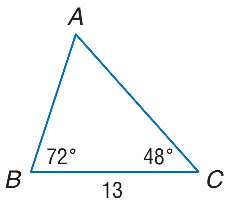
For Exercises	See Examples
10–27	1, 2
28–33	3

Extra Practice

See page 858.

Determine whether each triangle should be solved by beginning with the Law of Sines or Law of Cosines. Then solve each triangle. Round measures of sides to the nearest tenth and measures of angles to the nearest degree.

10.



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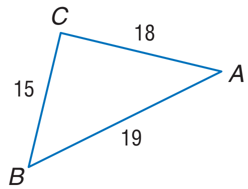

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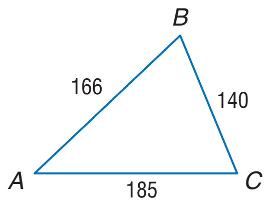

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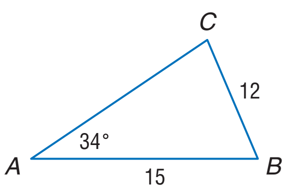

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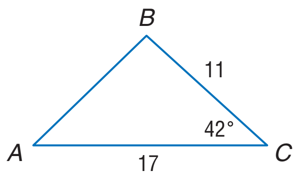

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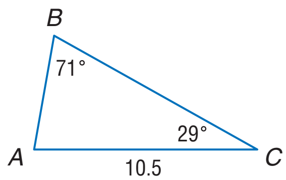


16. $a = 20$, $c = 24$, $B = 47°$

17. $a = 345$, $b = 648$, $c = 442$

18. $A = 36°$, $a = 10$, $b = 19$

19. $A = 25°$, $B = 78°$, $a = 13.7$

20. $a = 21.5$, $b = 16.7$, $c = 10.3$

21. $a = 16$, $b = 24$, $c = 41$

22. $a = 8$, $b = 24$, $c = 18$

23. $B = 19°$, $a = 51$, $c = 61$

24. $A = 56°$, $B = 22°$, $a = 12.2$

25. $a = 4$, $b = 8$, $c = 5$

26. $a = 21.5$, $b = 13$, $C = 38°$

27. $A = 40°$, $b = 7$, $c = 6$

CONTENTS

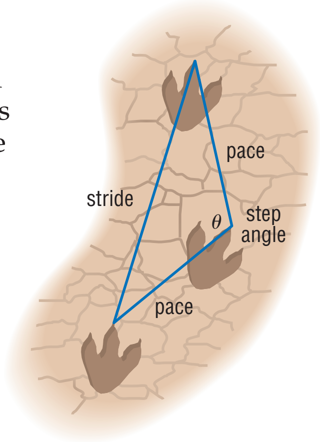
More About. . .

Dinosaurs

At digs such as the one at the Glen Rose formation in Texas, anthropologists study the footprints made by dinosaurs millions of years ago. *Locomoter* parameters, such as pace and stride, taken from these prints can be used to describe how a dinosaur once moved.

Source: Mid-America Paleontology Society

DINOSAURS **For Exercises 28–30, use the diagram at the right.**

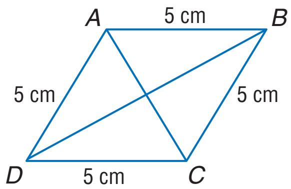


28. An anthropologist examining the footprints made by a bipedal (two-footed) dinosaur finds that the dinosaur's average pace was about 1.60 meters and average stride was about 3.15 meters. Find the step angle θ for this dinosaur.

29. Find the step angle θ made by the hindfeet of a herbivorous dinosaur whose pace averages 1.78 meters and stride averages 2.73 meters.

30. An efficient walker has a step angle that approaches 180°, meaning that the animal minimizes "zig-zag" motion while maximizing forward motion. What can you tell about the motion of each dinosaur from its step angle?

31. **GEOMETRY** In rhombus *ABCD*, the measure of $\angle ADC$ is 52°. Find the measures of diagonals $\overline{AC}$ and $\overline{DB}$ to the nearest tenth.



32. **SURVEYING** Two sides of a triangular plot of land have lengths of 425 feet and 550 feet. The measure of the angle between those sides is 44.5°. Find the perimeter and area of the plot.

33. **AVIATION** A pilot typically flies a route from Bloomington to Rockford, covering a distance of 117 miles. In order to avoid a storm, the pilot first flies from Bloomington to Peoria, a distance of 42 miles, then turns the plane and flies 108 miles on to Rockford. Through what angle did the pilot turn the plane over Peoria?

34. **CRITICAL THINKING** Explain how the Pythagorean Theorem is a special case of the Law of Cosines.

35. ***WRITING IN MATH*** Answer the question that was posed at the beginning of the lesson.

How can you determine the angle at which to install a satellite dish?

Include the following in your answer:

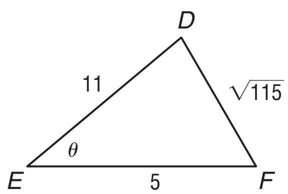
- a description of the conditions under which you can use the Law of Cosines to solve a triangle, and
- given the latitude of a point on Earth's surface, an explanation of how can you determine the angle at which to install a satellite dish at the same longitude.

Standardized Test Practice

36. In $\triangle DEF$, what is the value of θ to the nearest degree?

Ⓐ 26° Ⓑ 74°

Ⓒ 80° Ⓓ 141°

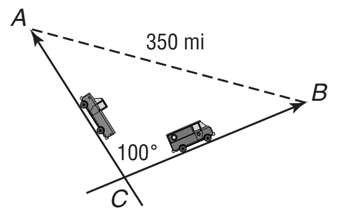


www.algebra2.com/self_check_quiz

CONTENTS

37. Two trucks, A and B, start from the intersection C of two straight roads at the same time. Truck A is traveling twice as fast as truck B and after 4 hours, the two trucks are 350 miles apart. Find the approximate speed of truck B in miles per hour.

(A) 35 (B) 37 (C) 57 (D) 73



Extending the Lesson

ERROR IN MEASUREMENT **For Exercises 38–40, use the following information.**
Consider $\triangle ABC$, in which $a = 17$, $b = 8$, and $c = 20$.

38. Find the measure of angle C in one step using the Law of Cosines. Round to the nearest tenth.

39. Find the measure of angle C in two steps using the Law of Cosines and then the Law of Sines. Round to the nearest tenth.

40. Explain why your answers for Exercises 38 and 39 are different. Which answer gives you the better approximation for the measure of angle C?

Check for Understanding

Mixed Review

Determine whether each triangle has *no* solution, *one* solution, or *two* solutions. Then solve each triangle. Round measures of sides to the nearest tenth and measures of angles to the nearest degree. *(Lesson 13-4)*

41. $A = 55°, a = 8, b = 7$

42. $A = 70°, a = 7, b = 10$

Find the exact values of the six trigonometric functions of θ if the terminal side of θ in standard position contains the given point. *(Lesson 13-3)*

43. $(5, 12)$

44. $(4, 7)$

45. $(\sqrt{10}, \sqrt{6})$

Solve each equation or inequality. *(Lesson 10-5)*

46. $e^x + 5 = 9$

47. $4e^x - 3 > -1$

48. $\ln(x + 3) = 2$

Getting Ready for the Next Lesson

PREREQUISITE SKILL **Find one angle with positive measure and one angle with negative measure coterminal with each angle.**
*(To review **coterminal angles**, see Lesson 13-2.)*

49. $45°$

50. $30°$

51. $180°$

52. $\frac{\pi}{2}$

53. $\frac{7\pi}{6}$

54. $\frac{4\pi}{3}$

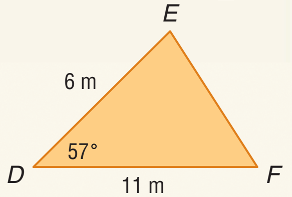
Practice Quiz 2

Lessons 13-3 through 13-5

1. Find the exact value of the six trigonometric functions of θ if the terminal side of θ in standard position contains the point $(-2, 3)$. *(Lesson 13-3)*

2. Find the exact value of $\csc \frac{5\pi}{3}$. *(Lesson 13-3)*

3. Find the area of $\triangle DEF$ to the nearest tenth. *(Lesson 13-4)*



4. Determine whether $\triangle ABC$, with $A = 22°$, $a = 15$, and $b = 18$, has *no* solution, *one* solution, or *two* solutions. Then solve the triangle, if possible. Round measures of sides to the nearest tenth and measures of angles to the nearest degree. *(Lesson 13-4)*

5. Determine whether $\triangle ABC$, with $b = 11$, $c = 14$, and $A = 78°$, should be solved by beginning with the Law of Sines or Law of Cosines. Then solve the triangle. Round measures of sides to the nearest tenth and measures of angles to the nearest degree. *(Lesson 13-5)*

13-6 Circular Functions

What You'll Learn

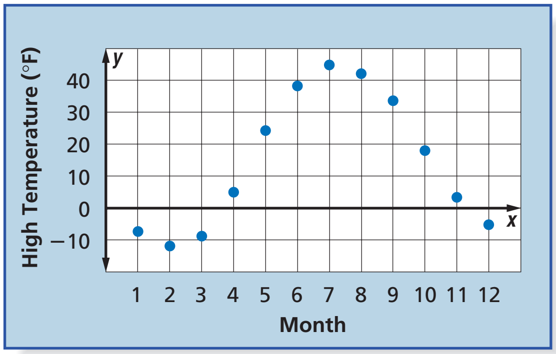
- Define and use the trigonometric functions based on the unit circle.
- Find the exact values of trigonometric functions of angles.

Vocabulary

- circular function
- periodic
- period

How can you model annual temperature fluctuations?

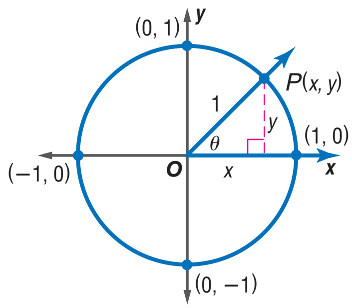
The average high temperatures, in degrees Fahrenheit, for Barrow, Alaska, are given in the table at the right. With January assigned a value of 1, February a value of 2, March a value of 3, and so on, these data can be graphed as shown below. This pattern of temperature fluctuations repeats after a period of 12 months.



MONTH	HIGH TEMP. (°F)
Jan	-7.4
Feb	-11.8
March	-9.0
April	4.7
May	24.2
June	38.3
July	45.0
Aug	42.3
Sept	33.8
Oct	18.1
Nov	3.5
Dec	-5.2

Source: www.met.utah.edu

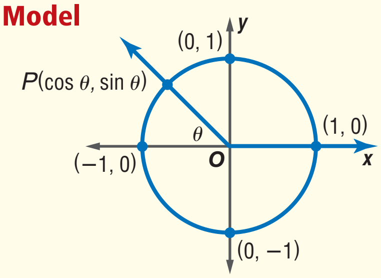
UNIT CIRCLE DEFINITIONS From your work with reference angles, you know that the values of trigonometric functions also repeat. For example, sin 30° and sin 150° have the same value, $\frac{1}{2}$. In this lesson, we will further generalize the trigonometric functions by defining them in terms of the unit circle.



Consider an angle θ in standard position. The terminal side of the angle intersects the unit circle at a unique point, $P(x, y)$. Recall that $\sin \theta = \frac{y}{r}$ and $\cos \theta = \frac{x}{r}$. Since $P(x, y)$ is on the unit circle, $r = 1$. Therefore, $\sin \theta = y$ and $\cos \theta = x$.

Key Concept — *Definition of Sine and Cosine*

- **Words** If the terminal side of an angle θ in standard position intersects the unit circle at $P(x, y)$, then $\cos \theta = x$ and $\sin \theta = y$. Therefore, the coordinates of P can be written as $P(\cos \theta, \sin \theta)$.
- **Model**



Since there is exactly one point $P(x, y)$ for any angle θ, the relations $\cos \theta = x$ and $\sin \theta = y$ are functions of θ. Because they are both defined using a unit circle, they are often called **circular functions**.

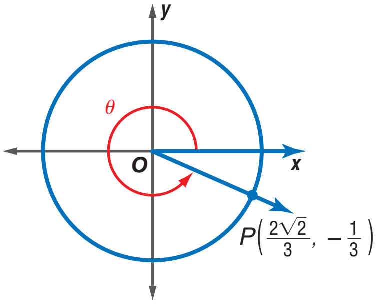
Study Tip

Reading Math

To help you remember that $x = \cos \theta$ and $y = \sin \theta$, notice that alphabetically x comes before y and cosine comes before sine.

Example 1 Find Sine and Cosine Given Point on Unit Circle

Given an angle θ in standard position, if $P\left(\frac{2\sqrt{2}}{3}, -\frac{1}{3}\right)$ lies on the terminal side and on the unit circle, find $\sin \theta$ and $\cos \theta$.



$$P\left(\frac{2\sqrt{2}}{3}, -\frac{1}{3}\right) = P(\cos \theta, \sin \theta),$$

so $\sin \theta = -\frac{1}{3}$ and $\cos \theta = \frac{2\sqrt{2}}{3}$.

In the Investigation below, you will explore the behavior of the sine and cosine functions on the unit circle.

Graphing Calculator Investigation

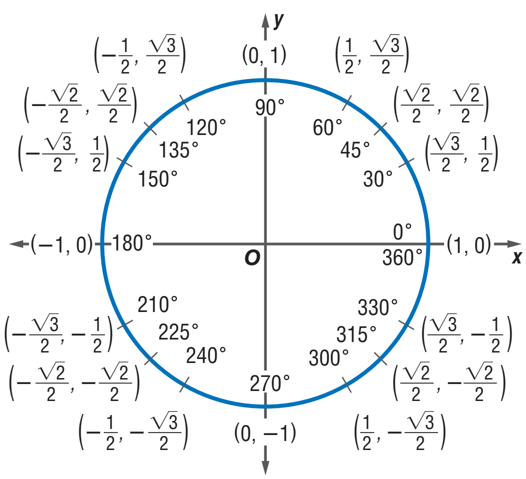
Sine and Cosine on the Unit Circle

Press MODE on a TI-83 Plus and highlight Degree and Par. Then use the following range values to set up a viewing window: TMIN = 0, TMAX = 360, TSTEP = 15, XMIN = −2.4, XMAX = 2.35, XSCL = 0.5, YMIN = −1.5, YMAX = 1.55, YSCL = 0.5. Press Y= to define the unit circle with $X_{1T} = \cos T$ and $Y_{1T} = \sin T$. Press GRAPH. Use the TRACE function to move around the circle.

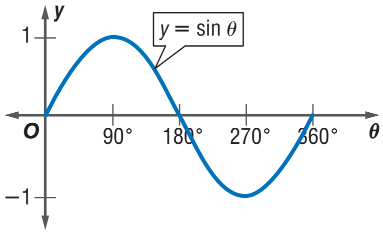
Think and Discuss

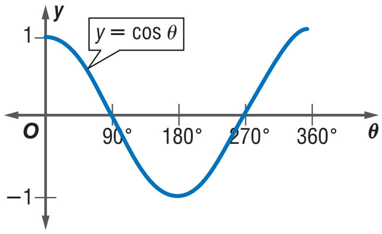
1. What does T represent? What does the x value represent? What does the y value represent?
2. Determine the sine and cosine of the angles whose terminal sides lie at 0°, 90°, 180°, and 270°.
3. How do the values of sine change as you move around the unit circle? How do the values of cosine change?

The exact values of the sine and cosine functions for specific angles are summarized using the definition of sine and cosine on the unit circle below.

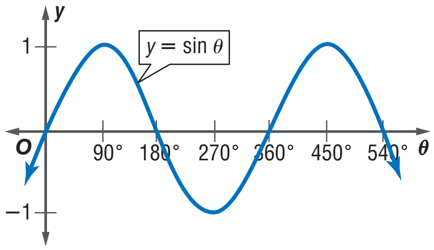


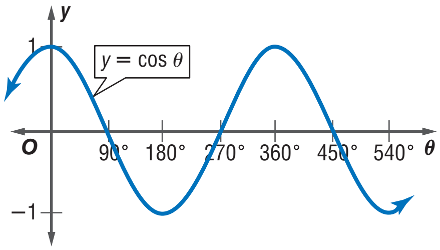
This same information is presented on the graphs of the sine and cosine functions below, where the horizontal axis shows the values of θ and the vertical axis shows the values of $\sin \theta$ or $\cos \theta$.





PERIODIC FUNCTIONS Notice in the graph above that the values of sine for the coterminal angles 0° and 360° are both 0. The values of cosine for these angles are both 1. Every 360° or 2π radians, the sine and cosine functions repeat their values. So, we can say that the sine and cosine functions are **periodic**, each having a **period** of 360° or 2π radians.





Key Concept — *Periodic Function*

A function is called periodic if there is a number a such that $f(x) = f(x + a)$ for all x in the domain of the function. The least positive value of a for which $f(x) = f(x + a)$ is called the period of the function.

For the sine and cosine functions, $\cos(x + 360°) = \cos x$, and $\sin(x + 360°) = \sin x$. In radian measure, $\cos(x + 2\pi) = \cos x$, and $\sin(x + 2\pi) = \sin x$. Therefore, the period of the sine and cosine functions is 360° or 2π.

Example 2 *Find the Value of a Trigonometric Function*

Find the exact value of each function.

a. $\cos 675°$

$$\begin{aligned}\cos 675° &= \cos(315° + 360°)\\ &= \cos 315°\\ &= \frac{\sqrt{2}}{2}\end{aligned}$$

b. $\sin\left(-\frac{5\pi}{6}\right)$

$$\begin{aligned}\sin\left(-\frac{5\pi}{6}\right) &= \sin\left(-\frac{5\pi}{6} + 2\pi\right)\\ &= \sin\frac{7\pi}{6}\\ &= -\frac{1}{2}\end{aligned}$$

When you look at the graph of a periodic function, you will see a repeating pattern: a shape that repeats over and over as you move to the right on the x-axis. The period is the distance along the x-axis from the beginning of the pattern to the point at which it begins again.

www.algebra2.com/extra_examples

CONTENTS

Many real-world situations have characteristics that can be described with periodic functions.

Example 3 Find the Value of a Trigonometric Function

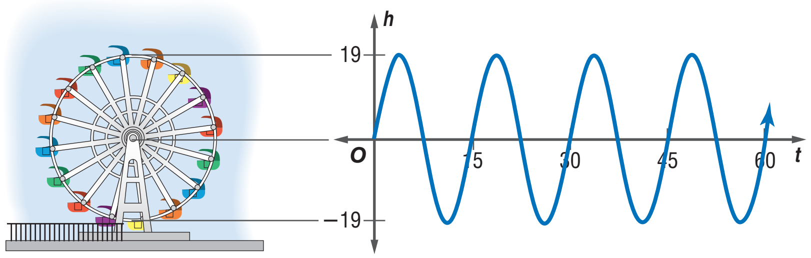
FERRIS WHEEL **As you ride a Ferris wheel, the height that you are above the ground varies periodically as a function of time. Consider the height of the center of the wheel to be the starting point. A particular wheel has a diameter of 38 feet and travels at a rate of 4 revolutions per minute.**

a. Identify the period of this function.

Since the wheel makes 4 complete counterclockwise rotations every minute, the period is the time it takes to complete one rotation, which is $\frac{1}{4}$ of a minute or 15 seconds.

b. Make a graph in which the horizontal axis represents the time *t* in seconds and the vertical axis represents the height *h* in feet in relation to the starting point.

Since the diameter of the wheel is 38 feet, the wheel reaches a maximum height of $\frac{38}{2}$ or 19 feet above the starting point and a minimum of 19 feet below the starting point.



Check for Understanding

Concept Check

1. **State** the conditions under which $\cos \theta = x$ and $\sin \theta = y$.
2. **OPEN ENDED** Give an example of a situation that could be described by a periodic function. Then state the period of the function.
3. **Compare and contrast** the graphs of the sine and cosine functions on page 741.

Guided Practice

If the given point P is located on the unit circle, find $\sin \theta$ and $\cos \theta$.

4. $P\left(\frac{5}{13}, -\frac{12}{13}\right)$

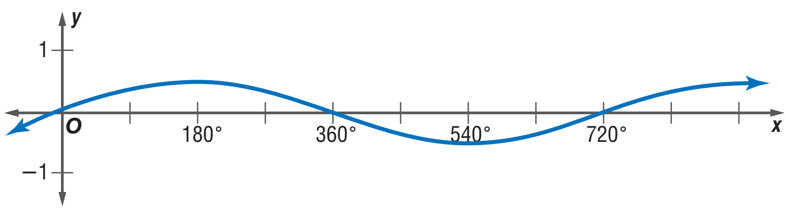
5. $P\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

Find the exact value of each function.

6. $\sin -240°$

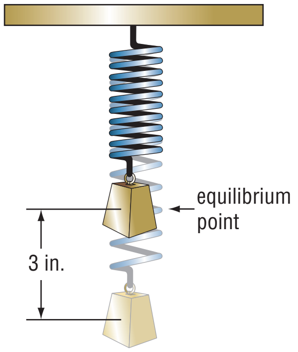
7. $\cos \frac{10\pi}{3}$

8. Determine the period of the function that is graphed below.



Application **PHYSICS** **For Exercises 9 and 10, use the following information.**

The motion of a weight on a spring varies periodically as a function of time. Suppose you pull the weight down 3 inches from its equilibrium point and then release it. It bounces above the equilibrium point and then returns below the equilibrium point in 2 seconds.



9. Find the period of this function.

10. Graph the height of the spring as a function of time.

Practice and Apply

Homework Help

For Exercises	See Examples
11–16	1
17–28	2
29–42	3

Extra Practice

See page 858.

The given point P is located on the unit circle. Find $\sin\theta$ and $\cos\theta$.

11. $P\left(-\frac{3}{5}, \frac{4}{5}\right)$ **12.** $P\left(-\frac{12}{13}, -\frac{5}{13}\right)$ **13.** $P\left(\frac{8}{17}, \frac{15}{17}\right)$

14. $P\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$ **15.** $P\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ **16.** $P(0.6, 0.8)$

Find the exact value of each function.

17. $\sin 690°$ **18.** $\cos 750°$ **19.** $\cos 5\pi$

20. $\sin\left(\frac{14\pi}{6}\right)$ **21.** $\sin\left(-\frac{3\pi}{2}\right)$ **22.** $\cos(-225°)$

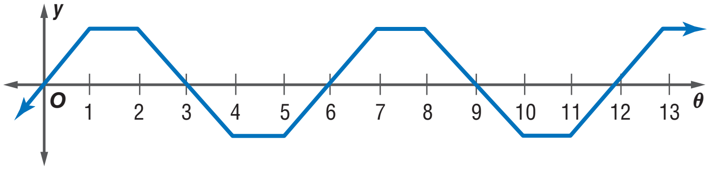
23. $\frac{\cos 60° + \sin 30°}{4}$ **24.** $3(\sin 60°)(\cos 30°)$

25. $\sin 30° - \sin 60°$ **26.** $\frac{4\cos 330° + 2\sin 60°}{3}$

27. $12(\sin 150°)(\cos 150°)$ **28.** $(\sin 30°)^2 + (\cos 30°)^2$

Determine the period of each function.

29.



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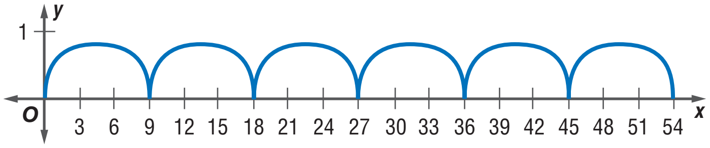

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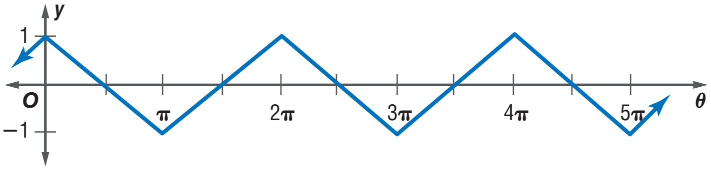

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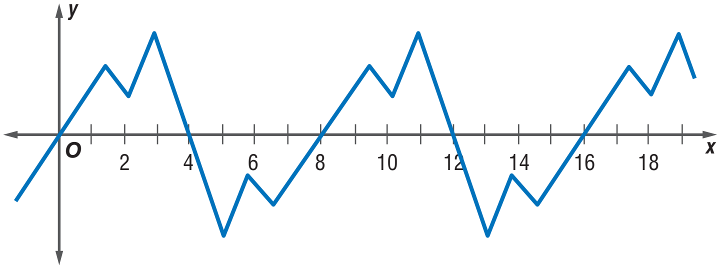


GUITAR **For Exercises 33 and 34, use the following information.**
When a guitar string is plucked, it is displaced from a fixed point in the middle of the string and vibrates back and forth, producing a musical tone. The exact tone depends on the frequency, or number of cycles per second, that the string vibrates. To produce an A, the frequency is 440 cycles per second, or 440 hertz (Hz).

33. Find the period of this function.

34. Graph the height of the fixed point on the string from its resting position as a function of time. Let the maximum distance above the resting position have a value of 1 unit and the minimum distance below this position have a value of 1 unit.

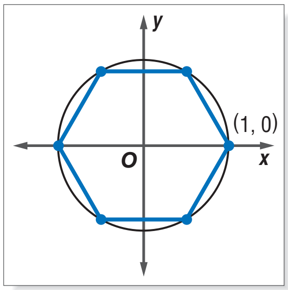
More About. . .

Guitar

Most guitars have six strings. The frequency at which one of these strings vibrates is controlled by the length of the string, the amount of tension on the string, the weight of the string, and springiness of the strings' material.

Source: www.howstuffworks.com

35. **GEOMETRY** A regular hexagon is inscribed in a unit circle centered at the origin. If one vertex of the hexagon is at (1, 0), find the exact coordinates of the remaining vertices.



36. **BIOLOGY** In a certain area of forested land, the population of rabbits R increases and decreases periodically throughout the year. If the population can be modeled by $R = 425 + 200 \sin\left[\frac{\pi}{365}(d - 60)\right]$, where d represents the dth day of the year, describe what happens to the population throughout the year.

SLOPE **For Exercises 37–42, use the following information.**
Suppose the terminal side of an angle θ in standard position intersects the unit circle at $P(x, y)$.

37. What is the slope of $\overline{OP}$?

38. Which of the six trigonometric functions is equal to the slope of $\overline{OP}$?

39. What is the slope of any line perpendicular to $\overline{OP}$?

40. Which of the six trigonometric functions is equal to the slope of any line perpendicular to $\overline{OP}$?

41. Find the slope of $\overline{OP}$ when $\theta = 60°$.

42. If $\theta = 60°$, find the slope of the line tangent to circle O at point P.

43. **CRITICAL THINKING** Determine the domain and range of the functions $y = \sin \theta$ and $y = \cos \theta$.

44. **WRITING IN MATH** Answer the question that was posed at the beginning of the lesson.

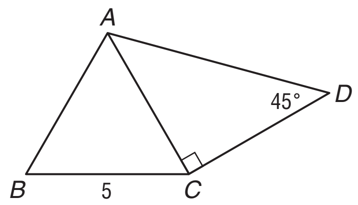
How can you model annual temperature fluctuations?

Include the following in your answer:

- a description of how the sine and cosine functions are similar to annual temperature fluctuations, and
- if the formula for the temperature T in degrees Fahrenheit of a city t months into the year is given by $T = 50 + 25 \sin\left(\frac{\pi}{6}t\right)$, explain how to find the average temperature and the maximum and minimum predicted over the year.

Standardized Test Practice

45. If $\triangle ABC$ is an equilateral triangle, what is the length of $\overline{AD}$, in units?



(A) $5\sqrt{2}$
(B) 5
(C) $10\sqrt{2}$
(D) 10

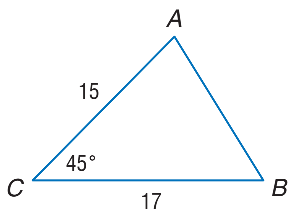
46. **SHORT RESPONSE** What is the exact value of tan 1830°?

Maintain Your Skills

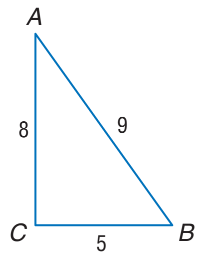
Mixed Review

Determine whether each triangle should be solved by beginning with the Law of Sines or Law of Cosines. Then solve each triangle. Round measures of sides to the nearest tenth and measures of angles to the nearest degree. *(Lesson 13-5)*

47.



48.



Find the area of $\triangle ABC$. Round to the nearest tenth. *(Lesson 13-4)*

49. $a = 11$ in., $c = 5$ in. , $B = 79°$

50. $b = 4$ m, $c = 7$ m, $A = 63°$

BULBS **For Exercises 51–56, use the following information.**
The lifetimes of 10,000 light bulbs are normally distributed. The mean lifetime is 300 days, and the standard deviation is 40 days. *(Lesson 12-7)*

51. How many light bulbs will last 260 and 340 days?

52. How many light bulbs will last between 220 and 380 days?

53. How many light bulbs will last fewer than 300 days?

54. How many light bulbs will last more than 300 days?

55. How many light bulbs will last more than 380 days?

56. How many light bulbs will last fewer than 180 days?

Find the sum of each infinite geometric series, if it exists. *(Lesson 11-5)*

57. $a_1 = 3, r = 1.2$

58. $16, 4, 1, \frac{1}{4}, \ldots$

59. $\sum_{n=1}^{\infty} 13(-0.625)^{n-1}$

Use synthetic division to find each quotient. *(Lesson 5-3)*

60. $(4x^2 - 13x + 10) \div (x - 2)$

61. $(2x^2 + 21x + 54) \div (x + 6)$

62. $(5y^3 + y^2 - 7) \div (y + 1)$

63. $(2y^2 + y - 16) \div (y - 3)$

Getting Ready for the Next Lesson

PREREQUISITE SKILL **Find each value of θ. Round to the nearest degree.**
*(To review **finding angle measures**, see Lesson 13-1.)*

64. $\sin \theta = 0.3420$

65. $\cos \theta = -0.3420$

66. $\tan \theta = 3.2709$

67. $\tan \theta = 5.6713$

68. $\sin \theta = 0.8290$

69. $\cos \theta = 0.0175$

13-7 Inverse Trigonometric Functions

What You'll Learn

- Solve equations by using inverse trigonometric functions.
- Find values of expressions involving trigonometric functions.

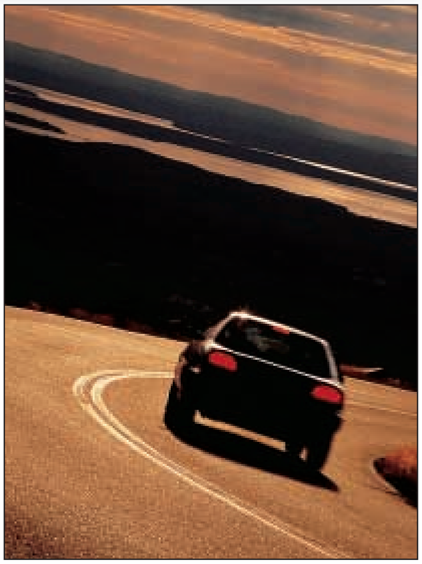
Vocabulary

- principal values
- Arcsine function
- Arccosine function
- Arctangent function

How are inverse trigonometric functions used in road design?

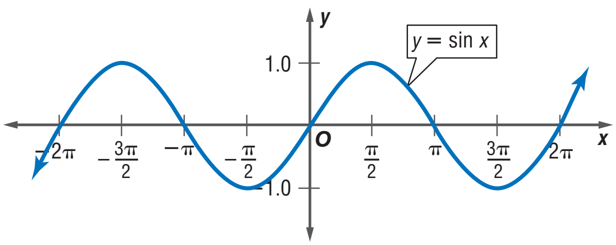
When a car travels a curve on a horizontal road, the friction between the tires and the road keeps the car on the road. Above a certain speed, however, the force of friction will not be great enough to hold the car in the curve. For this reason, civil engineers design banked curves.

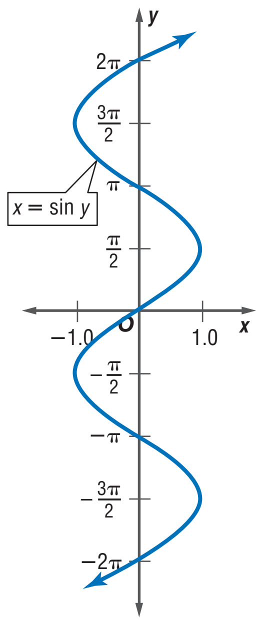
The proper banking angle θ for a car making a turn of radius r feet at a velocity v in feet per second is given by the equation $\tan \theta = \frac{v^2}{32r}$. In order to determine the appropriate value of θ for a specific curve, you need to know the radius of the curve, the maximum allowable velocity of cars making the curve, and how to determine the angle θ given the value of its tangent.



SOLVE EQUATIONS USING INVERSES Sometimes the value of a trigonometric function for an angle is known and it is necessary to find the measure of the angle. The concept of inverse functions can be applied to find the inverse of trigonometric functions.

In Lesson 8-8, you learned that the inverse of a function is the relation in which all the values of x and y are reversed. The graphs of $y = \sin x$ and its inverse, $x = \sin y$, are shown below.





Notice that the inverse is not a function, since it fails the vertical line test. None of the inverses of the trigonometric functions are functions.

We must restrict the domain of trigonometric functions so that their inverses are functions. The values in these restricted domains are called **principal values**. Capital letters are used to distinguish trigonometric functions with restricted domains from the usual trigonometric functions.

Key Concept — Principal Values of Sine, Cosine, and Tangent

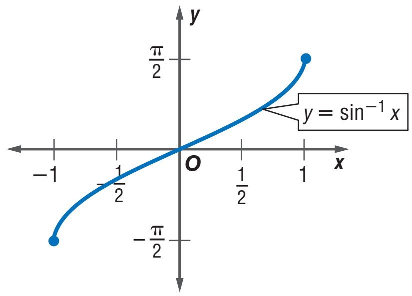
$y = \text{Sin } x$ if and only if $y = \sin x$ and $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$.

$y = \text{Cos } x$ if and only if $y = \cos x$ and $0 \leq x \leq \pi$.

$y = \text{Tan } x$ if and only if $y = \tan x$ and $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

The inverse of the Sine function is called the **Arcsine function** and is symbolized by **Sin^{-1}** or **Arcsin**. The Arcsine function has the following characteristics.

- Its domain is the set of real numbers from -1 to 1.
- Its range is the set of angle measures from $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$.
- $\text{Sin } x = y$ if and only if $\text{Sin}^{-1} y = x$.
- $[\text{Sin}^{-1} \circ \text{Sin}](x) = [\text{Sin} \circ \text{Sin}^{-1}](x) = x$.



Study Tip

Look Back
To review **composition of functions**, see Lesson 8-7.

The definitions of the Arccosine and Arctangent functions are similar to the definition of the Arcsine function.

Concept Summary — Inverse Sine, Cosine, and Tangent

- Given $y = \text{Sin } x$, the inverse Sine function is defined by $y = \text{Sin}^{-1} x$ or $y = \text{Arcsin } x$.
- Given $y = \text{Cos } x$, the inverse Cosine function is defined by $y = \text{Cos}^{-1} x$ or $y = \text{Arccos } x$.
- Given $y = \text{Tan } x$, the inverse Tangent function is defined by $y = \text{Tan}^{-1} x$ or $y = \text{Arctan } x$.

The expressions in each row of the table below are equivalent. You can use these expressions to rewrite and solve trigonometric equations.

$y = \text{Sin } x$	$x = \text{Sin}^{-1} y$	$x = \text{Arcsin } y$
$y = \text{Cos } x$	$x = \text{Cos}^{-1} y$	$x = \text{Arccos } y$
$y = \text{Tan } x$	$x = \text{Tan}^{-1} y$	$x = \text{Arctan } y$

Example 1 Solve an Equation

Solve $\text{Sin } x = \frac{\sqrt{3}}{2}$ by finding the value of x to the nearest degree.

If $\text{Sin } x = \frac{\sqrt{3}}{2}$, then x is the least value whose sine is $\frac{\sqrt{3}}{2}$. So, $x = \text{Arcsin } \frac{\sqrt{3}}{2}$.

Use a calculator to find x.

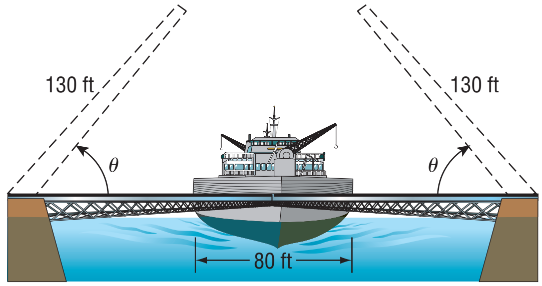
KEYSTROKES: 2nd [SIN^{-1}] 2nd [√] 3) ÷ 2) ENTER 60

Therefore, $x = 60°$.

Many application problems involve finding the inverse of a trigonometric function.

Example 2 Apply an Inverse to Solve a Problem

DRAWBRIDGE Each leaf of a certain double-leaf drawbridge is 130 feet long. If an 80-foot wide ship needs to pass through the bridge, what is the minimum angle θ, to the nearest degree, which each leaf of the bridge should open so that the ship will fit?



More About. . .

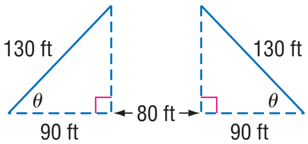
Drawbridges

Bascule bridges have spans (leaves) that pivot upward utilizing gears, motors, and counterweights.

Source: www.multnomah.lib.or.us

When the two parts of the bridge are in their lowered position, the bridge spans 130 + 130 or 260 feet. In order for the ship to fit, the distance between the leaves must be at least 80 feet.

This leaves a horizontal distance of $\frac{260 - 80}{2}$ or 90 feet from the pivot point of each leaf to the ship as shown in the diagram at the right.



To find the measure of angle θ, use the cosine ratio for right triangles.

$\cos\theta = \frac{\text{adj}}{\text{hyp}}$	Cosine ratio
$\cos\theta = \frac{90}{130}$	Replace *adj* with 90 and *hyp* with 130.
$\theta = \cos^{-1}\left(\frac{90}{130}\right)$	Inverse cosine function
$\theta \approx 46.2°$	Use a calculator.

Thus, the minimum angle through which each leaf of the bridge should open is 47°.

TRIGONOMETRIC VALUES You can use a calculator to find the values of trigonometric expressions.

Example 3 Find a Trigonometric Value

Find each value. Write angle measures in radians. Round to the nearest hundredth.

a. **ArcSin $\frac{\sqrt{3}}{2}$**

KEYSTROKES: [2nd] [SIN^{-1}] [2nd] [$\sqrt{\ }$] 3 [ )] [÷] 2 [ )] [ENTER] 1.047197551

Therefore, ArcSin $\frac{\sqrt{3}}{2} \approx 1.05$ radians.

b. **$\tan\left(\text{Cos}^{-1}\frac{6}{7}\right)$**

KEYSTROKES: [TAN] [2nd] [COS^{-1}] 6 [÷] 7 [ )] [ENTER] .6009252126

Therefore, $\tan\left(\text{Cos}^{-1}\frac{6}{7}\right) \approx 0.60$.

Study Tip

Angle Measure

Remember that when evaluating an inverse trigonometric function the result is an angle measure.

Check for Understanding

Concept Check

1. **Explain** how you know when the domain of a trigonometric function is restricted.
2. **OPEN ENDED** Write an equation giving the value of the Cosine function for an angle measure in its domain. Then, write your equation in the form of an inverse function.
3. Describe how $y = \text{Cos } x$ and $y = \text{Arccos } x$ are related.

Guided Practice

Write each equation in the form of an inverse function.

4. $\tan \theta = x$
5. $\cos \alpha = 0.5$

Solve each equation by finding the value of x to the nearest degree.

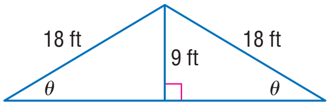
6. $x = \text{Cos}^{-1} \frac{\sqrt{2}}{2}$
7. $\text{Arctan } 0 = x$

Find each value. Write angle measures in radians. Round to the nearest hundredth.

8. $\text{Tan}^{-1}\left(-\frac{\sqrt{3}}{3}\right)$
9. $\text{Cos}^{-1}(-1)$
10. $\cos\left(\text{Cos}^{-1} \frac{2}{9}\right)$
11. $\sin\left(\text{Sin}^{-1} \frac{3}{4}\right)$
12. $\sin\left(\text{Cos}^{-1} \frac{3}{4}\right)$
13. $\tan\left(\text{Sin}^{-1} \frac{1}{2}\right)$

Application

14. **ARCHITECTURE** The support for a roof is shaped like two right triangles as shown at the right. Find θ.



Practice and Apply

Homework Help

For Exercises	See Examples
15–26	1
27–42	3
43–48	2

Extra Practice

See page 859.

Write each equation in the form of an inverse function.

15. $\alpha = \sin \beta$
16. $\tan a = b$
17. $\cos y = x$
18. $\sin 30° = \frac{1}{2}$
19. $\cos 45° = y$
20. $-\frac{4}{3} = \tan x$

Solve each equation by finding the value of x to the nearest degree.

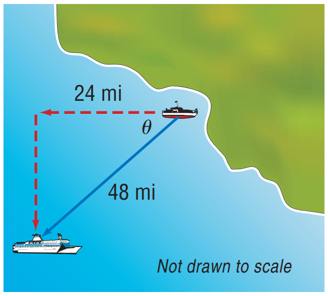
21. $x = \text{Cos}^{-1} \frac{1}{2}$
22. $\text{Sin}^{-1} \frac{1}{2} = x$
23. $\text{Arctan } 1 = x$
24. $x = \text{Arctan } \frac{\sqrt{3}}{3}$
25. $x = \text{Sin}^{-1} \frac{1}{\sqrt{2}}$
26. $x = \text{Cos}^{-1} 0$

Find each value. Write angle measures in radians. Round to the nearest hundredth.

27. $\text{Cos}^{-1}\left(-\frac{1}{2}\right)$
28. $\text{Sin}^{-1} \frac{\pi}{2}$
29. $\text{Arctan } \frac{\sqrt{3}}{3}$
30. $\text{Arccos } \frac{\sqrt{3}}{2}$
31. $\sin\left(\text{Sin}^{-1} \frac{1}{2}\right)$
32. $\cot\left(\text{Sin}^{-1} \frac{5}{6}\right)$
33. $\tan\left(\text{Cos}^{-1} \frac{6}{7}\right)$
34. $\sin\left(\text{Arctan } \frac{\sqrt{3}}{3}\right)$
35. $\cos\left(\text{Arcsin } \frac{3}{5}\right)$
36. $\cot\left(\text{Sin}^{-1} \frac{7}{9}\right)$
37. $\cos\left(\text{Tan}^{-1} \sqrt{3}\right)$
38. $\tan(\text{Arctan } 3)$
39. $\cos\left[\text{Arccos}\left(-\frac{1}{2}\right)\right]$
40. $\text{Sin}^{-1}\left(\tan \frac{\pi}{4}\right)$
41. $\cos\left(\text{Cos}^{-1} \frac{\sqrt{2}}{2} - \frac{\pi}{2}\right)$
42. $\cos^{-1}(\text{Sin}^{-1} 90)$
43. $\sin\left(2 \text{ Cos}^{-1} \frac{3}{5}\right)$
44. $\sin\left(2 \text{ Sin}^{-1} \frac{1}{2}\right)$

www.algebra2.com/self_check_quiz

45. **TRAVEL** The cruise ship *Reno* sailed due west 24 miles before turning south. When the *Reno* became disabled and radioed for help, the rescue boat found that the fastest route to her covered a distance of 48 miles. The cosine of the angle at which the rescue boat should sail is 0.5. Find the angle θ, to the nearest tenth of a degree, at which the rescue boat should travel to aid the *Reno*.



46. **FOUNTAINS** Architects who design fountains know that both the height and distance that a water jet will project is dependent on the angle θ at which the water is aimed. For a given angle θ, the ratio of the maximum height H of the parabolic arc to the horizontal distance D it travels is given by $\frac{H}{D} = \frac{1}{4}\tan\theta$. Find the value of θ, to the nearest degree, that will cause the arc to go twice as high as it travels horizontally.

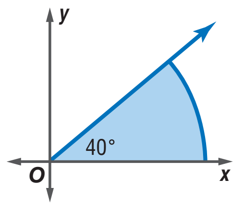
More About. . .

Track and Field

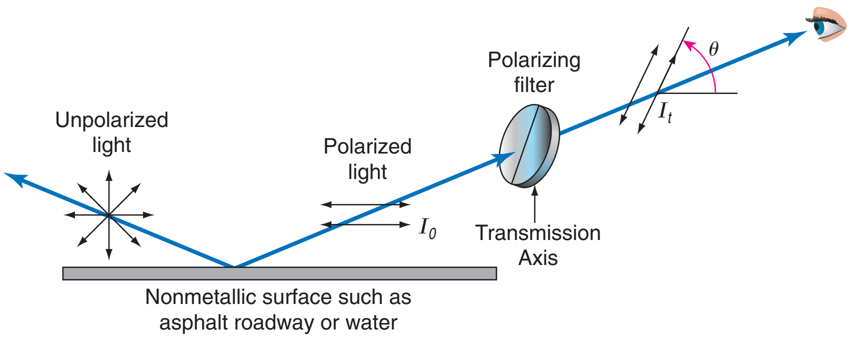
The shot is a metal sphere that can be made out of solid iron. Shot putters stand inside a seven-foot circle and must "put" the shot from the shoulder with one hand.

Source: www.coolrunning.com.au

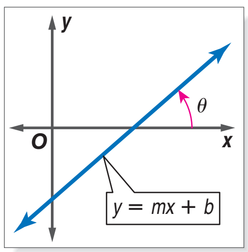
47. **TRACK AND FIELD** When a shot put is thrown, it must land in a 40° sector. Consider a coordinate system in which the vertex of the sector is at the origin and one side lies along the x-axis. If an athlete puts the shot so that it lands at a point with coordinates (18, 17), did the shot land in the required region? Explain your reasoning.



48. **OPTICS** You may have polarized sunglasses that eliminate glare by polarizing the light. When light is polarized, all of the waves are traveling in parallel planes. Suppose horizontally-polarized light with intensity I_0 strikes a polarizing filter with its axis at an angle of θ with the horizontal. The intensity of the transmitted light I_t and θ are related by the equation $\cos\theta = \sqrt{\frac{I_t}{I_0}}$. If one fourth of the polarized light is transmitted through the lens, what angle does the transmission axis of the filter make with the horizontal?



CRITICAL THINKING **For Exercises 49–51, use the following information.**
If the graph of the line $y = mx + b$ intersects the x-axis such that an angle of θ is formed with the positive x-axis, then $\tan\theta = m$.



49. Find the acute angle that the graph of $3x + 5y = 7$ makes with the positive x-axis to the nearest degree.

50. Determine the obtuse angle formed at the intersection of the graphs of $2x + 5y = 8$ and $6x - y = -8$. State the measure of the angle to the nearest degree.

51. Explain why this relationship, $\tan\theta = m$, holds true.

CONTENTS

52. **WRITING IN MATH** Answer the question that was posed at the beginning of the lesson.

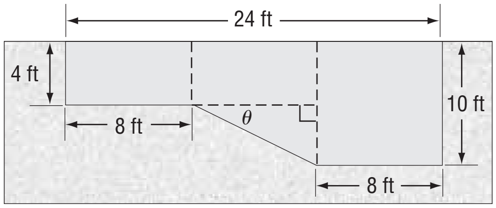
How are inverse trigonometric functions used in road design?

Include the following in your answer:

- a few sentences describing how to determine the banking angle for a road, and
- a description of what would have to be done to a road if the speed limit were increased and the banking angle was not changed.

Standardized Test Practice

53. **GRID IN** Find the angle of depression θ between the shallow end and the deep end of the swimming pool to the nearest degree.



Side View of Swimming Pool

54. If $\sin \theta = \frac{2}{3}$ and $-90° \leq \theta \leq 90°$, then $\cos 2\theta =$

Ⓐ $-\frac{1}{9}$. Ⓑ $-\frac{1}{3}$. Ⓒ $\frac{1}{3}$. Ⓓ $\frac{1}{9}$. Ⓔ 1.

Graphing Calculator

ADDITION OF TRIGONOMETRIC INVERSES Consider the function $y = \text{Sin}^{-1} x + \text{Cos}^{-1} x$.

55. Copy and complete the table below by evaluating y for each value of x.

x	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1
y									

56. Make a conjecture about the function $y = \text{Sin}^{-1} x + \text{Cos}^{-1} x$.

57. Considering only positive values of x, provide an explanation of why your conjecture might be true.

Maintain Your Skills

Mixed Review

Find the exact value of each function. *(Lesson 13-6)*

58. $\sin -660°$ 59. $\cos 25\pi$ 60. $(\sin 135°)^2 + (\cos -675°)^2$

Determine whether each triangle should be solved by beginning with the Law of Sines or Law of Cosines. Then solve each triangle. Round measures of sides to the nearest tenth and measures of angles to the nearest degree. *(Lesson 13-5)*

61. $a = 3.1$, $b = 5.8$, $A = 30°$ 62. $a = 9$, $b = 40$, $c = 41$

Use synthetic substitution to find $f(3)$ and $f(-4)$ for each function. *(Lesson 7-4)*

63. $f(x) = 5x^2 + 6x - 17$ 64. $f(x) = -3x^2 + 2x - 1$ 65. $f(x) = 4x^2 - 10x + 5$

66. **PHYSICS** A toy rocket is fired upward from the top of a 200-foot tower at a velocity of 80 feet per second. The height of the rocket t seconds after firing is given by the formula $h(t) = -16t^2 + 80t + 200$. Find the time at which the rocket reaches its maximum height of 300 feet. *(Lesson 6-5)*

Chapter 13 Study Guide and Review

Vocabulary and Concept Check

angle of depression (p. 705)
angle of elevation (p. 705)
arccosine function (p. 747)
arcsine function (p. 747)
arctangent function (p. 747)
circular function (p. 740)
cosecant (p. 701)
cosine (p. 701)
cotangent (p. 701)
coterminal angles (p. 712)
initial side (p. 709)
law of cosines (p. 733)
law of sines (p. 726)
period (p. 741)
periodic (p. 741)
principal values (p. 746)
quadrantal angles (p. 718)
radian (p. 710)
reference angle (p. 718)
secant (p. 701)
sine (p. 701)
solve a right triangle (p. 704)
standard position (p. 709)
tangent (p. 701)
terminal side (p. 709)
trigonometric functions (p. 701)
trigonometry (p. 701)
unit circle (p. 710)

State whether each sentence is *true* or *false*. If false, replace the underlined word(s) or number to make a true sentence.

1. When two angles in standard position have the same terminal side, they are called quadrantal angles.
2. The Law of Sines is used to solve a triangle when the measure of two angles and the measure of any side are known.
3. Trigonometric functions can be defined by using a unit circle.
4. For all values of θ, $\csc \theta = \frac{1}{\cos \theta}$.
5. A radian is the measure of an angle on the unit circle where the rays of the angle intercept an arc with length 1 unit.
6. If the measures of three sides of a triangle are known, then the Law of Sines can be used to solve the triangle.
7. An angle measuring $60°$ is a quadrantal angle.
8. For all values of x, $\cos(x + 180°) = \cos x$.
9. In a coordinate plane, the initial side of an angle is the ray that rotates about the center.

Lesson-by-Lesson Review

13-1 Right Triangle Trigonometry

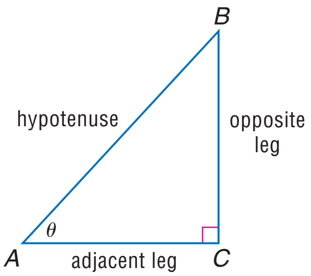
See pages 701–708.

Concept Summary

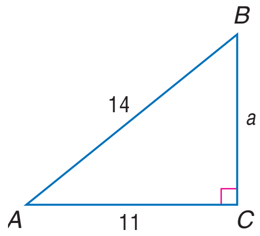
- If θ is the measure of an acute angle of a right triangle, *opp* is the measure of the leg opposite θ, *adj* is the measure of the leg adjacent to θ, and *hyp* is the measure of the hypotenuse, then the following are true.

$$\sin \theta = \frac{\text{opp}}{\text{hyp}} \quad \cos \theta = \frac{\text{adj}}{\text{hyp}} \quad \tan \theta = \frac{\text{opp}}{\text{adj}}$$

$$\csc \theta = \frac{\text{hyp}}{\text{opp}} \quad \sec \theta = \frac{\text{hyp}}{\text{adj}} \quad \cot \theta = \frac{\text{adj}}{\text{opp}}$$



Example **Solve $\triangle ABC$. Round measures of sides to the nearest tenth and measures of angles to the nearest degree.**



Find a.
$$a^2 + b^2 = c^2 \quad \text{Pythagorean Theorem}$$
$$a^2 + 11^2 = 14^2 \quad b = 11 \text{ and } c = 14$$
$$a = \sqrt{14^2 - 11^2} \quad \text{Solve for } a.$$
$$a \approx 8.7 \quad \text{Use a calculator.}$$

Find A. $\cos A = \frac{11}{14}$ $\quad \cos A = \frac{\text{adj}}{\text{hyp}}$

Use a calculator to find the angle whose cosine is $\frac{11}{14}$.

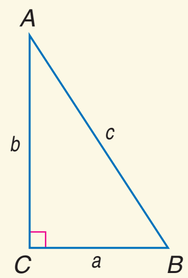
KEYSTROKES: 2nd [COS^{-1}] 11 ÷ 14) ENTER 38.2132107

To the nearest degree, $A \approx 38°$.

Find B.
$$38° + B \approx 90° \quad \text{Angles } A \text{ and } B \text{ are complementary.}$$
$$B \approx 52° \quad \text{Solve for } B.$$

Therefore, $a \approx 8.7$, $A \approx 38°$, and $B \approx 52°$.

Exercises **Solve $\triangle ABC$ by using the given measurements. Round measures of sides to the nearest tenth and measures of angles to the nearest degree.** *See Examples 4 and 5 on page 704.*



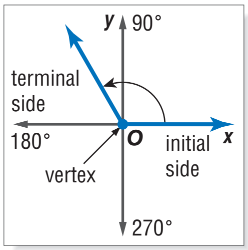
10. $c = 16, a = 7$
11. $A = 25°, c = 6$
12. $B = 45°, c = 12$
13. $B = 83°, b = \sqrt{31}$
14. $a = 9, B = 49°$
15. $\cos A = \frac{1}{4}, a = 4$

13-2 Angles and Angle Measure

See pages 709–715.

Concept Summary

- An angle in standard position has its vertex at the origin and its initial side along the positive x-axis.
- The measure of an angle is determined by the amount of rotation from the initial side to the terminal side. If the rotation is in a counterclockwise direction, the measure of the angle is positive. If the rotation is in a clockwise direction, the measure of the angle is negative.



Examples **Rewrite the degree measure in radians and the radian measure in degrees.**

1 **240°**
$$240° = 240°\left(\frac{\pi \text{ radians}}{180°}\right)$$
$$= \frac{240\pi}{180} \text{ radians or } \frac{4\pi}{3}$$

2 $\frac{\pi}{12}$
$$\frac{\pi}{12} = \left(\frac{\pi}{12} \text{ radians}\right)\left(\frac{180°}{\pi \text{ radians}}\right)$$
$$= \frac{180°}{12} \text{ or } 15°$$

Exercises **Rewrite each degree measure in radians and each radian measure in degrees.** *See Example 2 on page 711.*

16. $255°$ **17.** $-210°$ **18.** $\frac{7\pi}{4}$ **19.** -4π

Find one angle with positive measure and one angle with negative measure coterminal with each angle. *See Example 4 on page 712.*

20. $205°$ **21.** $-40°$ **22.** $\frac{4\pi}{3}$ **23.** $-\frac{7\pi}{4}$

13-3 Trigonometric Functions of General Angles

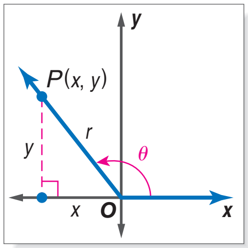
See pages 717–724.

Concept Summary

- You can find the exact values of the six trigonometric functions of θ given the coordinates of a point $P(x, y)$ on the terminal side of the angle.

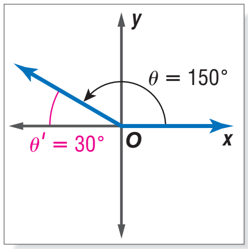
$\sin\theta = \frac{y}{r}$ $\cos\theta = \frac{x}{r}$ $\tan\theta = \frac{y}{x}, x \neq 0$

$\csc\theta = \frac{r}{y}, y \neq 0$ $\sec\theta = \frac{r}{x}, x \neq 0$ $\cot\theta = \frac{x}{y}, y \neq 0$



Example **Find the exact value of cos 150°.**

Because the terminal side of 150° lies in Quadrant II, the reference angle θ' is $180° - 150°$ or $30°$. The cosine function is negative in Quadrant II, so $\cos 150° = -\cos 30°$ or $-\frac{\sqrt{3}}{2}$.



Exercises **Find the exact value of the six trigonometric functions of θ if the terminal side of θ in standard position contains the given point.** *See Example 1 on pages 717 and 718.*

24. $P(2, 5)$ **25.** $P(15, -8)$

Find the exact value of each trigonometric function. *See Example 4 on page 720.*

26. $\cos 3\pi$ **27.** $\tan 120°$ **28.** $\sin\frac{5\pi}{4}$ **29.** $\sec(-30°)$

13-4 Law of Sines

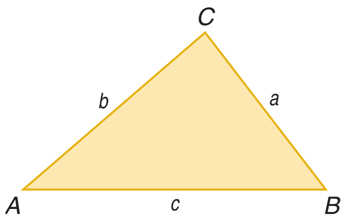
See pages 725–732.

Concept Summary

- You can find the area of $\triangle ABC$ if the measures of two sides and their included angle are known.

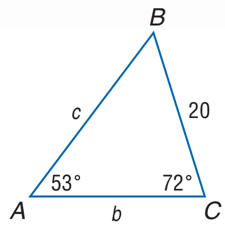
area $= \frac{1}{2}bc\sin A$ area $= \frac{1}{2}ac\sin B$ area $= \frac{1}{2}ab\sin C$

- Law of Sines: $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$



CONTENTS

Example **Solve $\triangle ABC$.**



First, find the measure of the third angle.

$53° + 72° + B = 180°$ The sum of the angle measures is 180°.

$B = 55°$ $180 - (53 + 72) = 55$

Now use the Law of Sines to find b and c. Write two equations, each with one variable.

$\frac{\sin A}{a} = \frac{\sin C}{c}$	Law of Sines	$\frac{\sin B}{b} = \frac{\sin A}{a}$
$\frac{\sin 53°}{20} = \frac{\sin 72°}{c}$	Replace A with 53°, B with 55°, C with 72°, and a with 20.	$\frac{\sin 55°}{b} = \frac{\sin 53°}{20}$
$c = \frac{20 \sin 72°}{\sin 53°}$	Solve for the variable.	$b = \frac{20 \sin 55°}{\sin 53°}$
$c \approx 23.8$	Use a calculator.	$b \approx 20.5$

Therefore, $B = 55°$, $b \approx 20.5$, and $c \approx 23.8$.

Exercises **Determine whether each triangle has *no* solution, *one* solution, or *two* solutions. Then solve each triangle. Round measures of sides to the nearest tenth and measures of angles to the nearest degree.** *See Examples 3–5 on pages 727 and 728.*

30. $a = 24, b = 36, A = 64°$

31. $A = 40°, b = 10, a = 8$

32. $b = 10, c = 15, C = 66°$

33. $A = 82°, a = 9, b = 12$

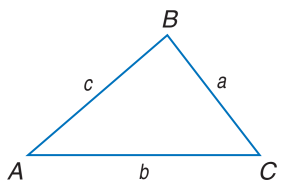
34. $A = 105°, a = 18, b = 14$

35. $B = 46°, C = 83°, b = 65$

13-5 Law of Cosines

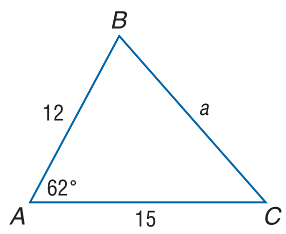
See pages 733–738.

Concept Summary



- Law of Cosines: $a^2 = b^2 + c^2 - 2bc \cos A$
  $b^2 = a^2 + c^2 - 2ac \cos B$
  $c^2 = a^2 + b^2 - 2ab \cos C$

Example **Solve $\triangle ABC$ for $A = 62°$, $b = 15$, and $c = 12$.**



You are given the measure of two sides and the included angle. Begin by drawing a diagram and using the Law of Cosines to determine a.

$a^2 = b^2 + c^2 - 2bc \cos A$ Law of Cosines

$a^2 = 15^2 + 12^2 - 2(15)(12) \cos 62°$ $b = 15$, $c = 12$, and $A = 62°$

$a^2 = 200$ Simplify.

$a \approx 14.1$ Take the square root of each side.

Next, you can use the Law of Sines to find the measure of angle C.

$\frac{\sin 62°}{14.1} \approx \frac{\sin C}{12}$ Law of Sines

$\sin C \approx \frac{12 \sin 62°}{14.1}$ or about 48.7° Use a calculator.

The measure of the angle B is approximately $180 - (62 + 48.7)$ or 69.3°.

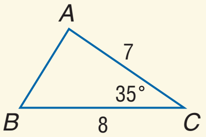
Therefore, $a \approx 14.1$, $C \approx 48.7°$, $B \approx 69.3°$.

Chapter 13 For More ...

- Extra Practice, see pages 857–859.
- Mixed Problem Solving, see page 874.

Exercises **Determine whether each triangle should be solved by beginning with the Law of Sines or Law of Cosines. Then solve each triangle. Round measures of sides to the nearest tenth and measures of angles to the nearest degree.** *See Examples 1 and 2 on pages 734 and 735.*

36.



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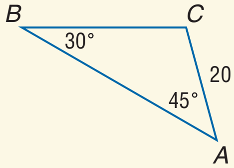

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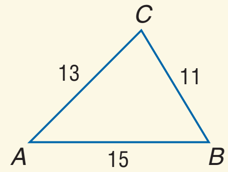


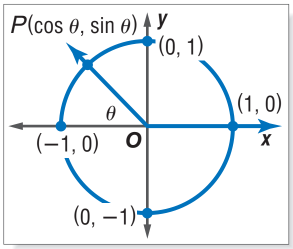
39. $C = 65°, a = 4, b = 7$ **40.** $A = 36°, a = 6, b = 8$ **41.** $b = 7.6, c = 14.1, A = 29°$

13-6 Circular Functions

See pages 739–745.

Concept Summary

- If the terminal side of an angle θ in standard position intersects the unit circle at $P(x, y)$, then $\cos \theta = x$ and $\sin \theta = y$. Therefore, the coordinates of P can be written as $P(\cos \theta, \sin \theta)$.



Example **Find the exact value of $\cos\left(-\frac{7\pi}{4}\right)$.**

$$\cos\left(-\frac{7\pi}{4}\right) = \cos\left(-\frac{7\pi}{4} + 2\pi\right) = \cos\frac{\pi}{4} \text{ or } \frac{\sqrt{2}}{2}$$

Exercises **Find the exact value of each function.** *See Example 2 on page 741.*

42. $\sin(-150°)$ **43.** $\cos 300°$ **44.** $(\sin 45°)(\sin 225°)$

45. $\sin\frac{5\pi}{4}$ **46.** $(\sin 30°)^2 + (\cos 30°)^2$ **47.** $\frac{4\cos 150° + 2\sin 300°}{3}$

13-7 Inverse Trigonometric Functions

See pages 746–751.

Concept Summary

- $y = \text{Sin}\ x$ if and only if $y = \sin x$ and $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$.
- $y = \text{Cos}\ x$ if and only if $y = \cos x$ and $0 \le x \le \pi$.
- $y = \text{Tan}\ x$ if and only if $y = \tan x$ and $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

Example **Find the value of $\text{Cos}^{-1}\left[\tan\left(-\frac{\pi}{6}\right)\right]$ in radians. Round to the nearest hundredth.**

KEYSTROKES: 2nd [COS^{-1}] TAN (–) 2nd [π] ÷ 6)) ENTER 2.186276035

Therefore, $\text{Cos}^{-1}\left[\tan\left(-\frac{\pi}{6}\right)\right] \approx 2.19$ radians.

Exercises **Find each value. Write angle measures in radians. Round to the nearest hundredth.** *See Example 3 on page 748.*

48. $\text{Sin}^{-1}(-1)$ **49.** $\text{Tan}^{-1}\sqrt{3}$ **50.** $\tan\left(\text{Arcsin}\ \frac{3}{5}\right)$ **51.** $\cos(\text{Sin}^{-1} 1)$

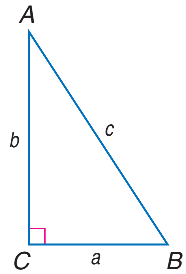
Chapter 13

Practice Test

Vocabulary and Concepts

1. **Draw** a right triangle and label one of the acute angles θ. Then label the hypotenuse *hyp*, the side opposite θ *opp*, and the side adjacent θ *adj*.
2. **State** the Law of Sines for $\triangle ABC$.
3. **Describe** a situation in which you would solve a triangle by first applying the Law of Cosines.

Skills and Applications



Solve $\triangle ABC$ by using the given measurements. Round measures of sides to the nearest tenth and measures of angles to the nearest degree.

4. $a = 7, A = 49°$
5. $B = 75°, b = 6$
6. $A = 22°, c = 8$
7. $a = 7, c = 16$

Rewrite each degree measure in radians and each radian measure in degrees.

8. $275°$
9. $-\frac{\pi}{6}$
10. $\frac{11\pi}{2}$
11. $330°$
12. $-600°$
13. $-\frac{7\pi}{4}$

Find the exact value of each expression. Write angle measures in degrees.

14. $\cos(-120°)$
15. $\sin \frac{7\pi}{4}$
16. $\cot 300°$
17. $\sec\left(-\frac{7\pi}{6}\right)$
18. $\mathrm{Sin}^{-1}\left(-\frac{\sqrt{3}}{2}\right)$
19. Arctan 1
20. $\tan 135°$
21. $\csc \frac{5\pi}{6}$

22. Determine the number of possible solutions for a triangle in which $A = 40°$, $b = 10$, and $a = 14$. If a solution exists, solve the triangle. Round measures of sides to the nearest tenth and measures of angles to the nearest degree.

23. Suppose θ is an angle in standard position whose terminal side lies in Quadrant II. Find the exact values of the remaining five trigonometric functions for θ for $\cos\theta = -\frac{\sqrt{3}}{2}$.

24. **GEOLOGY** From the top of the cliff, a geologist spots a dry riverbed. The measurement of the angle of depression to the riverbed is 70°. The cliff is 50 meters high. How far is the riverbed from the base of the cliff?

25. **STANDARDIZED TEST PRACTICE** Triangle *ABC* has a right angle at *C*, angle $B = 30°$, and $BC = 6$. Find the area of triangle *ABC*.

Ⓐ 6 units^2　Ⓑ $\sqrt{3}$ units^2　Ⓒ $6\sqrt{3}$ units^2　Ⓓ 12 units^2

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CONTENTS

Chapter 13 Standardized Test Practice

Part 1 Multiple Choice

Record your answers on the answer sheet provided by your teacher or on a sheet of paper.

1. If $3n + k = 30$ and n is a positive even integer, then which of the following statements must be true?
I. k is divisible by 3.
II. k is an even integer.
III. k is less than 20.

Ⓐ I only Ⓑ II only
Ⓒ I and II only Ⓓ I, II, and III

2. If $4x^2 + 5x = 80$ and $4x^2 - 5y = 30$, then what is the value of $6x + 6y$?

Ⓐ 10 Ⓑ 50 Ⓒ 60 Ⓓ 110

3. If $a = b + cb$, then what does $\frac{b}{a}$ equal in terms of c?

Ⓐ $\frac{1}{c}$ Ⓑ $\frac{1}{1 + c}$
Ⓒ $1 - c$ Ⓓ $1 + c$

4. What is the value of $\sum_{n=1}^{5} 3n^2$?

Ⓐ 55 Ⓑ 58
Ⓒ 75 Ⓓ 165

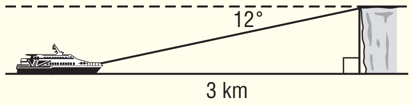
5. There are 16 green marbles, 2 red marbles, and 6 yellow marbles in a jar. How many yellow marbles need to be added to the jar in order to double the probability of selecting a yellow marble?

Ⓐ 4 Ⓑ 6 Ⓒ 8 Ⓓ 12

The Princeton Review **Test-Taking Tip**

Questions 1–10 The answer choices to multiple-choice questions can provide clues to help you solve a problem. In Question 5, you can add the values in the answer choices to the number of yellow marbles and the total number of marbles to find which is the correct answer.

6. From a lookout point on a cliff above a lake, the angle of depression to a boat on the water is 12°. The boat is 3 kilometers from the shore just below the cliff. What is the height of the cliff from the surface of the water to the lookout point?



Ⓐ $\frac{3}{\sin 12°}$ Ⓑ $\frac{3}{\tan 12°}$
Ⓒ $\frac{3}{\cos 12°}$ Ⓓ $3 \tan 12°$

7. If $x + y = 90°$ and x and y are positive, then $\frac{\cos x}{\sin y} =$

Ⓐ 0. Ⓑ $\frac{1}{2}$.
Ⓒ 1. Ⓓ cannot be determined

8. A child flying a kite holds the string 4 feet above the ground. The taut string is 40 feet long and makes an angle of 35° with the horizontal. How high is the kite off the ground?

Ⓐ $4 + 40 \sin 35°$ Ⓑ $4 + 40 \cos 35°$
Ⓒ $4 + 40 \tan 35°$ Ⓓ $4 + \frac{40}{\sin 35°}$

9. If $\sin \theta = -\frac{1}{2}$ and $180° < \theta < 270°$, then $\theta =$

Ⓐ 200°. Ⓑ 210°.
Ⓒ 225°. Ⓓ 240°.

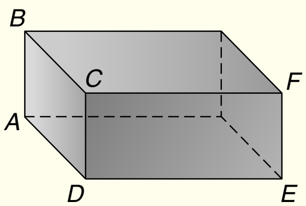
10. If $\cos \theta = \frac{8}{17}$ and the terminal side of the angle is in quadrant IV, then $\sin \theta =$

Ⓐ $-\frac{15}{8}$. Ⓑ $-\frac{17}{15}$.
Ⓒ $-\frac{15}{17}$. Ⓓ $\frac{15}{17}$.

Part 2 Short Response/Grid In

Record your answers on the answer sheet provided by your teacher or on a sheet of paper.

11. The length, width, and height of the rectangular box illustrated below are each integers greater than 1. If the area of *ABCD* is 18 square units and the area of *CDEF* is 21 square units, what is the volume of the box?

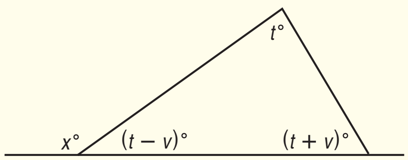


12. When six consecutive integers are multiplied, their product is 0. What is their greatest possible sum?

13. The average (arithmetic mean) score for the 25 players on a team is n. Their scores range from 60 to 100, inclusive. The average score of 20 of the players is 70. What is the difference between the greatest and least possible values of n?

14. The variables a, b, c, d, and e are integers in a sequence, where $a = 2$ and $b = 12$. To find the next term, double the last term and add that result to one less than the next-to-last term. For example, $c = 25$, because $2(12) = 24$, $2 - 1 = 1$, and $24 + 1 = 25$. What is the value of e?

15. In the figure, if $t = 2v$, what is the value of x?



16. If $b = 4$, then what is the value of a in the equations below?
$3a + 4b + 2c = 33$
$2b + 4c = 12$

17. At the head table at a banquet, 3 men and 3 women sit in a row. In how many ways can the row be arranged so that the men and women alternate?

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Part 3 Quantitative Comparison

Compare the quantity in Column A and the quantity in Column B. Then determine whether:

- Ⓐ **the quantity in Column A is greater,**
- Ⓑ **the quantity in Column B is greater,**
- Ⓒ **the two quantities are equal, or**
- Ⓓ **the relationship cannot be determined from the information given.**

Column A	Column B

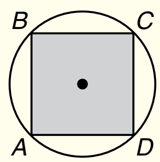
18. A container holds a certain number of tiles. The tiles are either red or white. One tile is chosen from the container at random.

probability of choosing a red or a white tile	200%

19. $\lozenge x = (4x)^4 + \frac{x}{4}$

5	$\lozenge \frac{1}{4}$

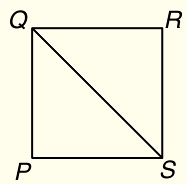
20.



The area of square *ABCD* is 64 units^2.

area of circle *O*	192 units^2

21. *PQRS* is a square.



$\frac{QS}{RS}$	2