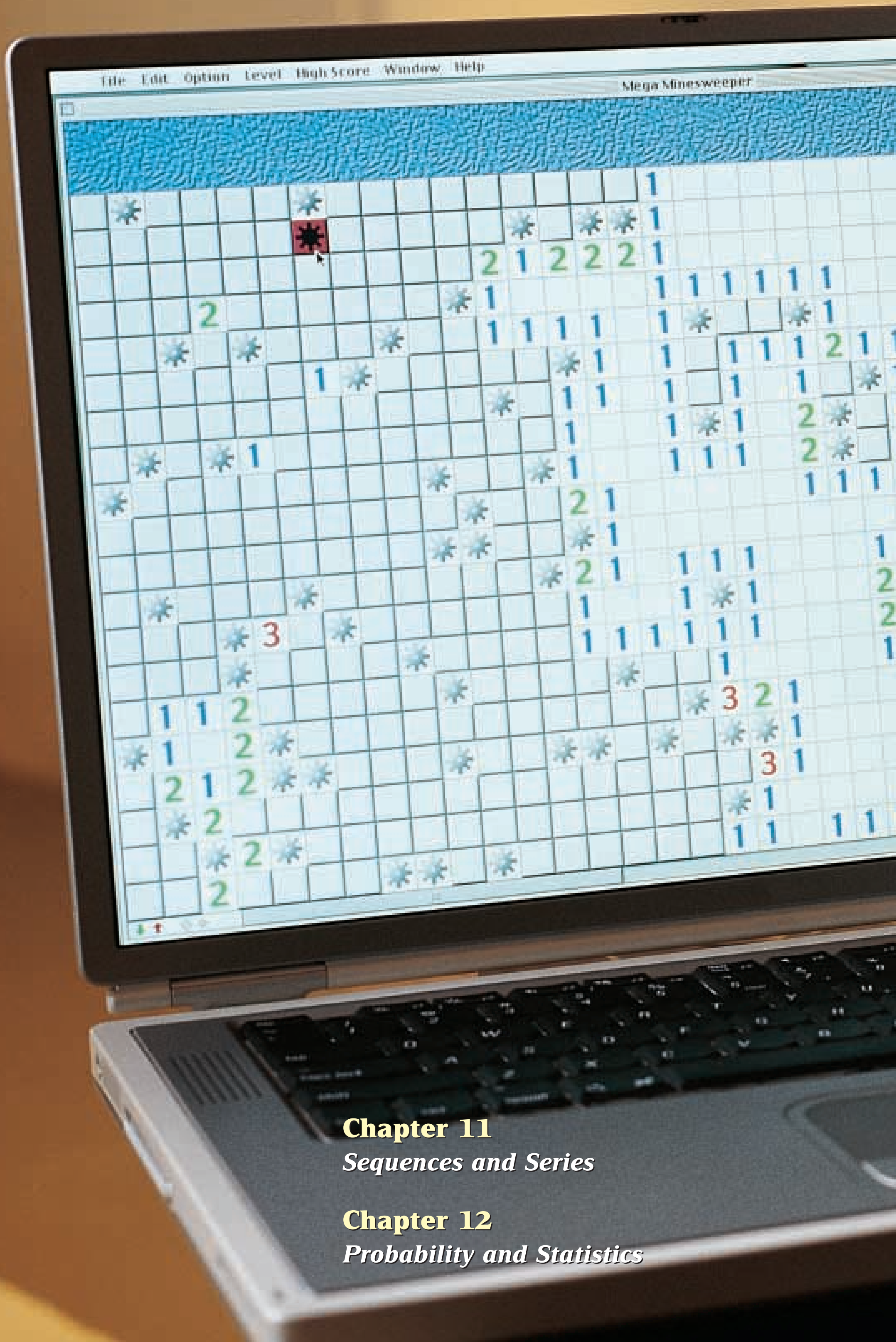
UNIT 4

Discrete Mathematics

Discrete mathematics is the branch of mathematics that involves finite or discontinuous quantities. In this unit, you will learn about sequences, series, probability, and statistics.

Richard Kaye
Professor of Mathematics
University of Birmingham



Chapter 11
Sequences and Series

Chapter 12
Probability and Statistics

WebQuest Internet Project

'Minesweeper': Secret to Age-Old Puzzle?

Source: *USA TODAY,* November 3, 2000

"*Minesweeper*, a seemingly simple game included on most personal computers, could help mathematicians crack one of the field's most intriguing problems. The buzz began after Richard Kaye, a mathematics professor at the University of Birmingham in England, started playing *Minesweeper*. After playing the game steadily for a few weeks, Kaye realized that *Minesweeper*, if played on a much larger grid, has the same mathematical characteristics as other problems deemed insolvable." In this project, you will research a mathematician of the past and his or her role in the development of discrete mathematics.

Log on to www.algebra2.com/webquest. Begin your WebQuest by reading the Task.

Then continue working on your WebQuest as you study Unit 4.

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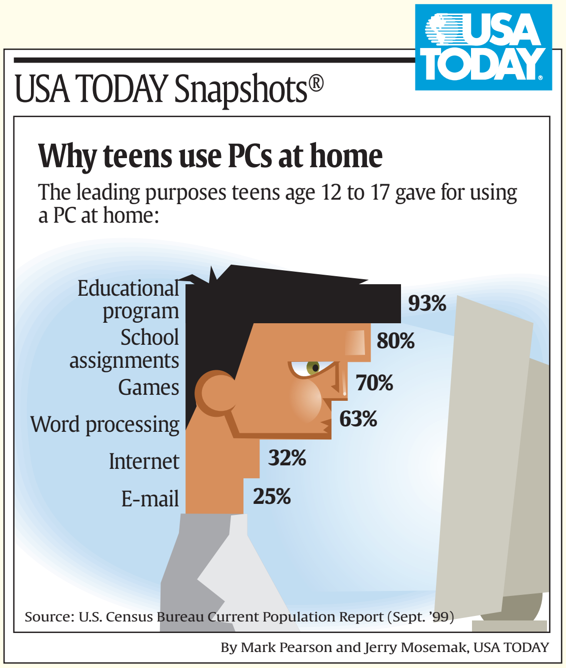


Chapter

11 Sequences and Series

What You'll Learn

- **Lessons 11-1 through 11-5** Use arithmetic and geometric sequences and series.
- **Lesson 11-6** Use special sequences and iterate functions.
- **Lesson 11-7** Expand powers by using the Binomial Theorem.
- **Lesson 11-8** Prove statements by using mathematical induction.

Key Vocabulary

- **arithmetic sequence** (p. 578)
- **arithmetic series** (p. 583)
- **sigma notation** (p. 585)
- **geometric sequence** (p. 588)
- **geometric series** (p. 594)

Why It's Important

Many number patterns found in nature and used in business can be modeled by sequences, which are lists of numbers. Some sequences are classified by the method used to predict the next term from the previous term(s). When the terms of a sequence are added, a series is formed. *In Lesson 11-2, you will learn how the number of seats in the rows of an amphitheater can be modeled using a series.*

CONTENTS

Getting Started

Prerequisite Skills To be successful in this chapter, you'll need to master these skills and be able to apply them in problem-solving situations. Review these skills before beginning Chapter 11.

For Lessons 11-1 and 11-3 — Solve Equations

Solve each equation. *(For review, see Lessons 1-3 and 5-5.)*

1. $36 = 12 + 4x$

2. $-40 = 10 + 5x$

3. $12 - 3x = 27$

4. $162 = 2x^4$

5. $\frac{1}{8} = 4x^5$

6. $3x^3 + 4 = -20$

For Lessons 11-1 and 11-5 — Graph Functions

Graph each function. *(For review, see Lesson 2-1.)*

7. $\{(1, 1), (2, 3), (3, 5), (4, 7), (5, 9)\}$

8. $\{(1, -20), (2, -16), (3, -12), (4, -8), (5, -4)\}$

9. $\left\{(1, 64), (2, 16), (3, 4), (4, 1), \left(5, \frac{1}{4}\right)\right\}$

10. $\left\{(1, 2), (2, 3), \left(3, \frac{7}{2}\right), \left(4, \frac{15}{4}\right), \left(5, \frac{31}{8}\right)\right\}$

For Lessons 11-1 through 11-5, 11-8 — Evaluate Expressions

Evaluate each expression for the given value(s) of the variable(s). *(For review, see Lesson 1-1.)*

11. $x + (y - 1)z$ if $x = 3$, $y = 8$, and $z = 2$

12. $\frac{x}{2}(y + z)$ if $x = 10$, $y = 3$, and $z = 25$

13. $a \cdot b^{c-1}$ if $a = 2$, $b = \frac{1}{2}$, and $c = 7$

14. $\frac{a(1 - bc)^2}{1 - b}$ if $a = -2$, $b = 3$, and $c = 5$

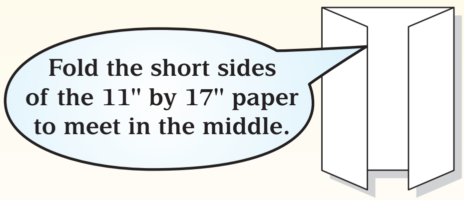
15. $\frac{a}{1 - b}$ if $a = \frac{1}{2}$, and $b = \frac{1}{6}$

16. $\frac{n(n + 1)}{2}$ if $n = 10$

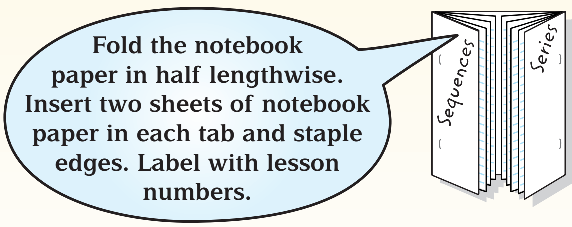
FOLDABLES™ Study Organizer

Make this Foldable to record information about sequences and series. Begin with one sheet of 11" by 17" paper and four sheets of notebook paper.

Step 1 Fold and Cut



Step 2 Staple and Label



Reading and Writing As you read and study the chapter, fill the journal with examples for each lesson.

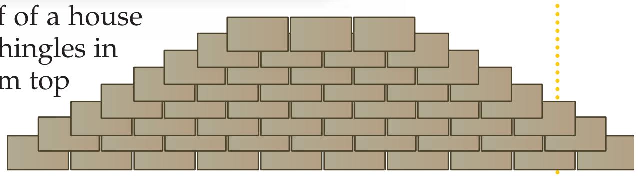
11-1 Arithmetic Sequences

What You'll Learn

- Use arithmetic sequences.
- Find arithmetic means.

How are arithmetic sequences related to roofing?

A roofer is nailing shingles to the roof of a house in overlapping rows. There are three shingles in the top row. Since the roof widens from top to bottom, one additional shingle is needed in each successive row.

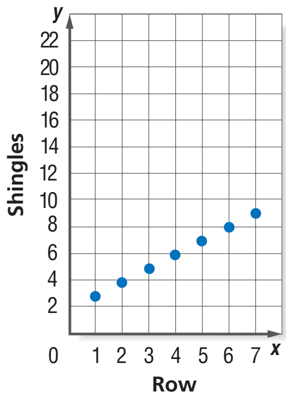


Row	1	2	3	4	5	6	7
Shingles	3	4	5	6	7	8	9

Vocabulary

- sequence
- term
- arithmetic sequence
- common difference
- arithmetic means

ARITHMETIC SEQUENCES The numbers 3, 4, 5, 6, …, representing the number of shingles in each row, are an example of a sequence of numbers. A **sequence** is a list of numbers in a particular order. Each number in a sequence is called a **term**. The first term is symbolized by a_1, the second term is symbolized by a_2, and so on.



The graph represents the information from the table above. A sequence is a function whose domain is the set of positive integers. You can see from the graph that a sequence is a discrete function.

Many sequences have patterns. For example, in the sequence above for the number of shingles, each term can be found by adding 1 to the previous term. A sequence of this type is called an arithmetic sequence. An **arithmetic sequence** is a sequence in which each term after the first is found by adding a constant, called the **common difference** d, to the previous term.

Study Tip

Sequences
The numbers in a sequence may not be ordered. For example, the numbers 33, 25, 36, 40, 36, 66, 63, 50, ... are a sequence that represents the number of home runs Sammy Sosa hit in each year beginning with 1993.

Example 1 Find the Next Terms

Find the next four terms of the arithmetic sequence 55, 49, 43, … .

Find the common difference d by subtracting two consecutive terms.

$49 - 55 = -6$ and $43 - 49 = -6$ So, $d = -6$.

Now add -6 to the third term of the sequence, and then continue adding -6 until the next four terms are found.

$$43 \xrightarrow{+(-6)} 37 \xrightarrow{+(-6)} 31 \xrightarrow{+(-6)} 25 \xrightarrow{+(-6)} 19$$

The next four terms of the sequence are 37, 31, 25, and 19.

There is a pattern in the way the terms of an arithmetic sequence are formed. It is possible to develop a formula for each term of an arithmetic sequence in terms of the first term a_1 and the common difference d. Look at the sequence in Example 1.

Sequence	numbers	55	49	43	37	…	
	symbols	a_1	a_2	a_3	a_4	…	a_n
Expressed in Terms of d and the First Term	numbers	$55 + 0(-6)$	$55 + 1(-6)$	$55 + 2(-6)$	$55 + 3(-6)$	…	$55 + (n - 1)(-6)$
	symbols	$a_1 + 0 \cdot d$	$a_1 + 1 \cdot d$	$a_1 + 2 \cdot d$	$a_1 + 3 \cdot d$	…	$a_1 + (n - 1)d$

The following formula generalizes this pattern for any arithmetic sequence.

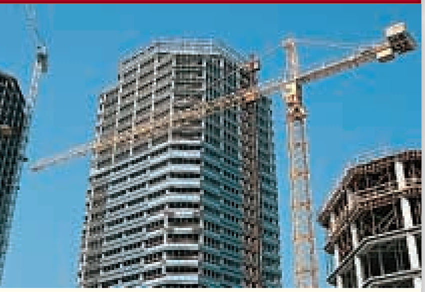
Key Concept — *n*th Term of an Arithmetic Sequence

The *n*th term a_n of an arithmetic sequence with first term a_1 and common difference d is given by

$$a_n = a_1 + (n - 1)d,$$

where n is any positive integer.

More About. . .



Construction

The table below shows typical costs for a construction company to rent a crane for one, two, three, or four months.

Months	Cost ($)
1	75,000
2	90,000
3	105,000
4	120,000

Source: www.howstuffworks.com

Example 2 Find a Particular Term

CONSTRUCTION Refer to the information at the left. Assuming that the arithmetic sequence continues, how much would it cost to rent the crane for twelve months?

Explore Since the difference between any two successive costs is $15,000, the costs form an arithmetic sequence with common difference 15,000.

Plan You can use the formula for the *n*th term of an arithmetic sequence with $a_1 = 75{,}000$ and $d = 15{,}000$ to find a_{12}, the cost for twelve months.

Solve

$a_n = a_1 + (n - 1)d$ — Formula for *n*th term

$a_{12} = 75{,}000 + (12 - 1)15{,}000$ — $n = 12$, $a_1 = 75{,}000$, $d = 15{,}000$

$a_{12} = 240{,}000$ — Simplify.

It would cost $240,000 to rent the crane for twelve months.

Examine You can find terms of the sequence by adding 15,000. a_5 through a_{12} are 135,000, 150,000, 165,000, 180,000, 195,000, 210,000, 225,000, and 240,000. Therefore, $240,000 is correct.

Example 3 Write an Equation for the *n*th Term

Write an equation for the *n*th term of the arithmetic sequence 8, 17, 26, 35, … .

In this sequence, $a_1 = 8$ and $d = 9$. Use the *n*th term formula to write an equation.

$a_n = a_1 + (n - 1)d$ — Formula for *n*th term

$a_n = 8 + (n - 1)9$ — $a_1 = 8$, $d = 9$

$a_n = 8 + 9n - 9$ — Distributive Property

$a_n = 9n - 1$ — Simplify.

An equation is $a_n = 9n - 1$.

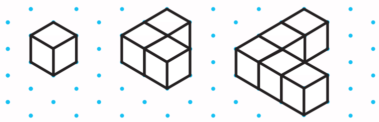
Study Tip

Arithmetic Sequences
An equation for an arithmetic sequence is always linear.

Algebra Activity

Arithmetic Sequences

Study the figures below. The length of an edge of each cube is 1 centimeter.



Model and Analyze

1. Based on the pattern, draw the fourth figure on a piece of isometric dot paper.
2. Find the volumes of the four figures.
3. Suppose the number of cubes in the pattern continues. Write an equation that gives the volume of Figure *n*.
4. What would the volume of the twelfth figure be?

ARITHMETIC MEANS Sometimes you are given two terms of a sequence, but they are not successive terms of that sequence. The terms between any two nonsuccessive terms of an arithmetic sequence are called **arithmetic means**. In the sequence below, 41, 52, and 63 are the three arithmetic means between 30 and 74.

19, 30, 41, 52, 63, 74, 85, 96, …

3 arithmetic means between 30 and 74

Example 4 Find Arithmetic Means

Find the four arithmetic means between 16 and 91.

You can use the *n*th term formula to find the common difference. In the sequence 16, __?__, __?__, __?__, __?__, 91, …, a_1 is 16 and a_6 is 91.

$a_n = a_1 + (n - 1)d$	Formula for the *n*th term
$a_6 = 16 + (6 - 1)d$	$n = 6, a_1 = 16$
$91 = 16 + 5d$	$a_6 = 91$
$75 = 5d$	Subtract 16 from each side.
$15 = d$	Divide each side by 5.

Now use the value of *d* to find the four arithmetic means.

16 → (+ 15) → 31 → (+ 15) → 46 → (+ 15) → 61 → (+ 15) → 76

The arithmetic means are 31, 46, 61, and 76. **CHECK** $76 + 15 = 91$ ✓

Study Tip

Alternate Method

You may prefer this method. The four means will be $16 + d$, $16 + 2d$, $16 + 3d$, and $16 + 4d$. The common difference is $d = 91 - (16 + 4d)$ or $d = 15$.

Check for Understanding

Concept Check

1. **Explain** why the sequence 4, 5, 7, 10, 14, … is not arithmetic.
2. **Find** the 15th term in the arithmetic sequence −3, 4, 11, 18, … .
3. **OPEN ENDED** Write an arithmetic sequence with common difference −5.

Guided Practice

Find the next four terms of each arithmetic sequence.

4. 12, 16, 20, …
5. 3, 1, −1, …

Find the first five terms of each arithmetic sequence described.

6. $a_1 = 5, d = 3$
7. $a_1 = 14, d = -2$

8. Find a_{13} for the arithmetic sequence $-17, -12, -7, \ldots$.

Find the indicated term of each arithmetic sequence.

9. $a_1 = 3, d = -5, n = 24$

10. $a_1 = -5, d = 7, n = 13$

11. Complete: 68 is the __?__ th term of the arithmetic sequence $-2, 3, 8, \ldots$.

12. Write an equation for the nth term of the arithmetic sequence $-26, -15, -4, 7, \ldots$.

13. Find the three arithmetic means between 44 and 92.

Application

14. **ENTERTAINMENT** A basketball team has a halftime promotion where a fan gets to shoot a 3-pointer to try to win a jackpot. The jackpot starts at \$5000 for the first game and increases \$500 each time there is no winner. Ken has tickets to the fifteenth game of the season. How much will the jackpot be for that game if no one wins by then?

Practice and Apply

Homework Help

For Exercises	See Examples
15–28, 49	1
29–45, 51	2
46–48, 50	3
52–55	4

Extra Practice

See page 851.

Find the next four terms of each arithmetic sequence.

15. $9, 16, 23, \ldots$

16. $31, 24, 17, \ldots$

17. $-6, -2, 2, \ldots$

18. $-8, -5, -2, \ldots$

19. $\frac{1}{3}, 1, \frac{5}{3}, \ldots$

20. $\frac{18}{5}, \frac{16}{5}, \frac{14}{5}, \ldots$

21. $6.7, 6.3, 5.9, \ldots$

22. $1.3, 3.8, 6.3, \ldots$

Find the first five terms of each arithmetic sequence described.

23. $a_1 = 2, d = 13$

24. $a_1 = 41, d = 5$

25. $a_1 = 6, d = -4$

26. $a_1 = 12, d = -3$

27. $a_1 = \frac{4}{3}, d = -\frac{1}{3}$

28. $a_1 = \frac{5}{8}, d = \frac{3}{8}$

29. Find a_8 if $a_n = 4 + 3n$.

30. If $a_n = 1 - 5n$, what is a_{10}?

Find the indicated term of each arithmetic sequence.

31. $a_1 = 3, d = 7, n = 14$

32. $a_1 = -4, d = -9, n = 20$

33. $a_1 = 35, d = 3, n = 101$

34. $a_1 = 20, d = 4, n = 81$

35. $a_1 = 5, d = \frac{1}{3}, n = 12$

36. $a_1 = \frac{5}{2}, d = -\frac{3}{2}, n = 11$

37. a_{12} for $-17, -13, -9, \ldots$

38. a_{12} for $8, 3, -2, \ldots$

39. a_{21} for $121, 118, 115, \ldots$

40. a_{43} for $5, 9, 13, 17, \ldots$

41. **GEOLOGY** Geologists estimate that the continents of Europe and North America are drifting apart at a rate of an average of 12 miles every 1 million years, or about 0.75 inch per year. If the continents continue to drift apart at that rate, how many inches will they drift in 50 years? (*Hint*: $a_1 = 0.75$)

More About. . .

Tower of Pisa

Upon its completion in 1370, the Leaning Tower of Pisa leaned about 1.7 meters from vertical. Today, it leans about 5.2 meters from vertical.

Source: Associated Press

42. **TOWER OF PISA** To prove that objects of different weights fall at the same rate, Galileo dropped two objects with different weights from the Leaning Tower of Pisa in Italy. The objects hit the ground at the same time. When an object is dropped from a tall building, it falls about 16 feet in the first second, 48 feet in the second second, and 80 feet in the third second, regardless of its weight. How many feet would an object fall in the tenth second?

Complete the statement for each arithmetic sequence.

43. 170 is the __?__ term of $-4, 2, 8, \ldots$.

44. 124 is the __?__ term of $-2, 5, 12, \ldots$.

45. -14 is the __?__ term of $2\frac{1}{5}, 2, 1\frac{4}{5}, \ldots$.

Write an equation for the *n*th term of each arithmetic sequence.

46. $7, 16, 25, 34, \ldots$

47. $18, 11, 4, -3, \ldots$

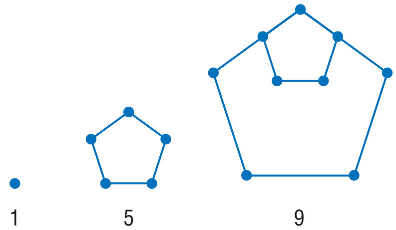
48. $-3, -5, -7, -9, \ldots$

GEOMETRY **For Exercises 49–51, refer to the first three arrays of numbers below.**

49. Make drawings to find the next three numbers in this pattern.

50. Write an equation representing the *n*th number in this pattern.

51. Is 397 a number in this pattern? Explain.



Find the arithmetic means in each sequence.

52. 55, __?__, __?__, __?__, 115

53. 10, __?__, __?__, -8

54. -8, __?__, __?__, __?__, __?__, 7

55. 3, __?__, __?__, __?__, __?__, __?__, 27

56. **CRITICAL THINKING** The numbers x, y, and z are the first three terms of an arithmetic sequence. Express z in terms of x and y.

57. **WRITING IN MATH** Answer the question that was posed at the beginning of the lesson.

How are arithmetic sequences related to roofing?

Include the following in your answer:

- the words that indicate that the numbers of shingles in the rows form an arithmetic sequence, and
- explanations of at least two ways to find the number of shingles in the fifteenth row.

Standardized Test Practice

58. What number follows 20 in this arithmetic sequence?

$8, 11, 14, 17, 20, \ldots$

(A) 5 (B) 23 (C) 26 (D) 29

59. Find the first term in the arithmetic sequence.

_____, $8\frac{1}{3}, 7, 5\frac{2}{3}, 4\frac{1}{3}, \ldots$

(A) 3 (B) $9\frac{2}{3}$ (C) $10\frac{1}{3}$ (D) 11

Maintain Your Skills

Mixed Review

60. **COMPUTERS** Suppose a computer that costs \$3000 new is only worth \$600 after 3 years. What is the average annual rate of depreciation? *(Lesson 10-6)*

Solve each equation. *(Lesson 10-5)*

61. $3e^x - 2 = 0$

62. $e^{3x} = 4$

63. $\ln(x + 2) = 5$

64. If y varies directly as x and $y = 5$ when $x = 2$, find y when $x = 6$. *(Lesson 9-4)*

Getting Ready for the Next Lesson

PREREQUISITE SKILL **Evaluate each expression for the given values of the variable.** *(To review **evaluating expressions**, see Lesson 1-1.)*

65. $3n - 1; n = 1, 2, 3, 4$

66. $6 - j; j = 1, 2, 3, 4$

67. $4m + 7; m = 1, 2, 3, 4, 5$

11-2 Arithmetic Series

What You'll Learn

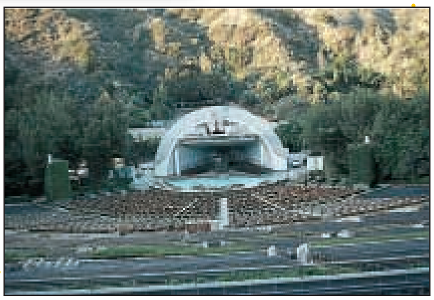
- Find sums of arithmetic series.
- Use sigma notation.

Vocabulary

- series
- arithmetic series
- sigma notation
- index of summation

How do arithmetic series apply to amphitheaters?

The first amphitheaters were built for contests between gladiators. Modern amphitheaters are usually used for the performing arts. Amphitheaters generally get wider as the distance from the stage increases. Suppose a small amphitheater can seat 18 people in the first row and each row can seat 4 more people than the previous row.



ARITHMETIC SERIES The numbers of seats in the rows of the amphitheater form an arithmetic sequence. To find the number of people who could sit in the first four rows, add the first four terms of the sequence. That sum is 18 + 22 + 26 + 30 or 96. A **series** is an indicated sum of the terms of a sequence. Since 18, 22, 26, 30 is an arithmetic sequence, 18 + 22 + 26 + 30 is an **arithmetic series**. Below are some more arithmetic sequences and the corresponding arithmetic series.

Study Tip

Indicated Sum

The sum of a series is the result when the terms of the series are added. An *indicated sum* is the expression that illustrates the series, which includes the terms + or −.

Arithmetic Sequence	Arithmetic Series
5, 8, 11, 14, 17	5 + 8 + 11 + 14 + 17
−9, −3, 3	−9 + (−3) + 3
$\frac{3}{8}, \frac{8}{8}, \frac{13}{8}, \frac{18}{8}$	$\frac{3}{8} + \frac{8}{8} + \frac{13}{8} + \frac{18}{8}$

S_n represents the sum of the first n terms of a series. For example, S_4 is the sum of the first four terms. For the series 5 + 8 + 11 + 14 + 17, S_4 is 5 + 8 + 11 + 14 or 38.

To develop a formula for the sum of any arithmetic series, consider the series below.

$$S_9 = 4 + 11 + 18 + 25 + 32 + 39 + 46 + 53 + 60$$

Suppose we write S_9 in two different orders and add the two equations.

$$\begin{aligned} S_9 &= 4 + 11 + 18 + 25 + 32 + 39 + 46 + 53 + 60 \\ (+)\ S_9 &= 60 + 53 + 46 + 39 + 32 + 25 + 18 + 11 + 4 \\ \hline 2S_9 &= 64 + 64 + 64 + 64 + 64 + 64 + 64 + 64 + 64 \\ 2S_9 &= 9(64) \\ S_9 &= \frac{9}{2}(64) \end{aligned}$$

Note that the sum had 9 terms.

The first and last terms of the sum are 64.

An arithmetic sequence S_n has n terms, and the sum of the first and last terms is $a_1 + a_n$. Thus, the formula $S_n = \frac{n}{2}(a_1 + a_n)$ represents the sum of any arithmetic series.

Key Concept — Sum of an Arithmetic Series

The sum S_n of the first n terms of an arithmetic series is given by

$$S_n = \frac{n}{2}[2a_1 + (n - 1)d] \text{ or } S_n = \frac{n}{2}(a_1 + a_n).$$

Example 1 Find the Sum of an Arithmetic Series

Find the sum of the first 100 positive integers.

The series is $1 + 2 + 3 + \ldots + 100$. Since you can see that $a_1 = 1$, $a_{100} = 100$, and $d = 1$, you can use either sum formula for this series.

Method 1

$S_n = \frac{n}{2}(a_1 + a_n)$ Sum formula

$S_{100} = \frac{100}{2}(1 + 100)$ $n = 100$, $a_1 = 1$, $a_{100} = 100$, $d = 1$

$S_{100} = 50(101)$ Simplify.

$S_{100} = 5050$ Multiply.

Method 2

$S_n = \frac{n}{2}[2a_1 + (n - 1)d]$

$S_{100} = \frac{100}{2}[2(1) + (100 - 1)1]$

$S_{100} = 50(101)$

$S_{100} = 5050$

The sum of the first 100 positive integers is 5050.

Example 2 Find the First Term

RADIO **A radio station considered giving away $4000 every day in the month of August for a total of $124,000. Instead, they decided to increase the amount given away every day while still giving away the same total amount. If they want to increase the amount by $100 each day, how much should they give away the first day?**

You know the values of n, S_n, and d. Use the sum formula that contains d.

$S_n = \frac{n}{2}[2a_1 + (n - 1)d]$ Sum formula

$S_{31} = \frac{31}{2}[2a_1 + (31 - 1)100]$ $n = 31$, $d = 100$

$124{,}000 = \frac{31}{2}(2a_1 + 3000)$ $S_{31} = 124{,}000$

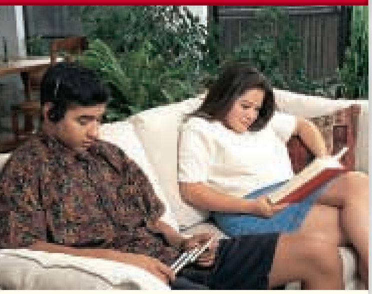
$8000 = 2a_1 + 3000$ Multiply each side by $\frac{2}{31}$.

$5000 = 2a_1$ Subtract 3000 from each side.

$2500 = a_1$ Divide each side by 2.

The radio station should give away $2500 the first day.

More About. . .



Radio

99.0% of teens ages 12–17 listen to the radio at least once a week. 79.1% listen at least once a day.

Source: Radio Advertising Bureau

Sometimes it is necessary to use both a sum formula and the formula for the nth term to solve a problem.

Example 3 Find the First Three Terms

Find the first three terms of an arithmetic series in which $a_1 = 9$, $a_n = 105$, and $S_n = 741$.

Step 1 Since you know a_1, a_n, and S_n, use $S_n = \frac{n}{2}(a_1 + a_n)$ to find n.

$S_n = \frac{n}{2}(a_1 + a_n)$

$741 = \frac{n}{2}(9 + 105)$

$741 = 57n$

$13 = n$

Step 2 Find d.

$a_n = a_1 + (n - 1)d$

$105 = 9 + (13 - 1)d$

$96 = 12d$

$8 = d$

Step 3 Use d to determine a_2 and a_3.

$a_2 = 9 + 8$ or 17 $a_3 = 17 + 8$ or 25

The first three terms are 9, 17, and 25.

Study Tip

Sigma Notation

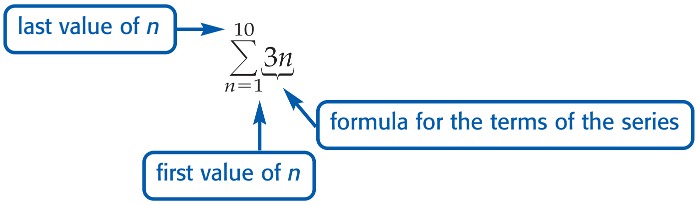
There are many ways to represent a given series.

$\sum_{r=4}^{9}(r-3)$

$=\sum_{s=2}^{7}(s-1)$

$=\sum_{j=0}^{5}(j+1)$

SIGMA NOTATION Writing out a series can be time-consuming and lengthy. For convenience, there is a more concise notation called **sigma notation**. The series $3 + 6 + 9 + 12 + \ldots + 30$ can be expressed as $\sum_{n=1}^{10} 3n$. This expression is read *the sum of 3n as n goes from 1 to 10.*



The variable, in this case n, is called the **index of summation**.

To generate the terms of a series given in sigma notation, successively replace the index of summation with consecutive integers between the first and last values of the index, inclusive. For the series above, the values of n are 1, 2, 3, and so on, through 10.

Example 4 Evaluate a Sum in Sigma Notation

Evaluate $\sum_{j=5}^{8}(3j-4)$.

Method 1

Find the terms by replacing j with 5, 6, 7, and 8. Then add.

$$\sum_{j=5}^{8}(3j-4) = [3(5)-4] + [3(6)-4] + [3(7)-4] + [3(8)-4]$$
$$= 11 + 14 + 17 + 20$$
$$= 62$$

Method 2

Since the sum is an arithmetic series, use the formula $S_n = \frac{n}{2}(a_1 + a_n)$.

There are 4 terms, $a_1 = 3(5) - 4$ or 11, and $a_4 = 3(8) - 4$ or 20.

$$S_4 = \frac{4}{2}(11 + 20)$$
$$S_4 = 62$$

The sum of the series is 62.

You can use the sum and sequence features on a graphing calculator to find the sum of a series.

Study Tip

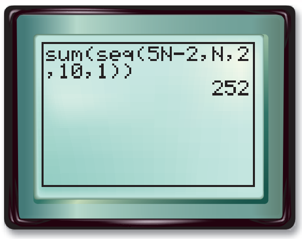
Graphing Calculators

On the TI-83 Plus, sum(is located on the LIST MATH menu. The function seq(is located on the LIST OPS menu.

Graphing Calculator Investigation

Sums of Series

The calculator screen shows the evaluation of $\sum_{N=2}^{10}(5N-2)$. The first four entries for seq(are



- the formula for the general term of the series,
- the index of summation,
- the first value of the index, and
- the last value of the index, respectively.

The last entry is always 1 for the types of series that we are considering.

Think and Discuss

1. Explain why you can use any letter for the index of summation.

2. Evaluate $\sum_{n=1}^{8}(2n-1)$ and $\sum_{j=5}^{12}(2j-9)$. **Make a conjecture** as to their relationship and explain why you think it is true.

www.algebra2.com/extra_examples

Check for Understanding

Concept Check

1. **Explain** the difference between a sequence and a series.
2. **OPEN ENDED** Write an arithmetic series for which $S_5 = 10$.
3. **OPEN ENDED** Write the series $7 + 10 + 13 + 16$ using sigma notation.

Guided Practice

Find S_n for each arithmetic series described.

4. $a_1 = 4, a_n = 100, n = 25$
5. $a_1 = 40, n = 20, d = -3$
6. $a_1 = 132, d = -4, a_n = 52$
7. $d = 5, n = 16, a_n = 72$

Find the sum of each arithmetic series.

8. $5 + 11 + 17 + \ldots + 95$
9. $38 + 35 + 32 + \ldots + 2$
10. $\sum_{n=1}^{7} (2n + 1)$
11. $\sum_{k=3}^{7} (3k + 4)$

Find the first three terms of each arithmetic series described.

12. $a_1 = 11, a_n = 110, S_n = 726$
13. $n = 8, a_n = 36, S_n = 120$

Application

14. **WORLD CULTURES** The African-American festival of *Kwanzaa* includes a ritual involving candles. The first night, a candle is lit and then blown out. The second night, a new candle and the candle from the previous night are lit and blown out. This pattern of lighting a new candle and relighting all the candles from the previous nights is continued for seven nights. Use a formula from this lesson to find the total number of candle lightings during the festival.

Practice and Apply

Homework Help

For Exercises	See Examples
15–32, 39, 40, 45	1, 2
33–38	4
41–44	3

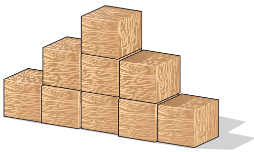
Extra Practice

See page 851.

Find S_n for each arithmetic series described.

15. $a_1 = 7, a_n = 79, n = 8$
16. $a_1 = 58, a_n = -7, n = 26$
17. $a_1 = 43, n = 19, a_n = 115$
18. $a_1 = 76, n = 21, a_n = 176$
19. $a_1 = 7, d = -2, n = 9$
20. $a_1 = 3, d = -4, n = 8$
21. $a_1 = 5, d = \frac{1}{2}, n = 13$
22. $a_1 = 12, d = \frac{1}{3}, n = 13$
23. $d = -3, n = 21, a_n = -64$
24. $d = 7, n = 18, a_n = 72$
25. $d = \frac{1}{5}, n = 10, a_n = \frac{23}{10}$
26. $d = -\frac{1}{4}, n = 20, a_n = -\frac{53}{12}$

27. **TOYS** Jamila is making a triangular wall with building blocks. The top row has one block, the second row has three, the third has five, and so on. How many rows can she make with a set of 100 blocks?



28. **CONSTRUCTION** A construction company will be fined for each day it is late completing its current project. The daily fine will be \$4000 for the first day and will increase by \$1000 each day. Based on its budget, the company can only afford \$60,000 in total fines. What is the maximum number of days it can be late?

Find the sum of each arithmetic series.

29. $6 + 13 + 20 + 27 + \ldots + 97$
30. $7 + 14 + 21 + 28 + \ldots + 98$
31. $34 + 30 + 26 + \ldots + 2$
32. $16 + 10 + 4 + \ldots + (-50)$
33. $\sum_{n=1}^{6} (2n + 11)$
34. $\sum_{n=1}^{5} (2 - 3n)$
35. $\sum_{k=7}^{11} (42 - 9k)$
36. $\sum_{t=19}^{23} (5t - 3)$
37. $\sum_{i=1}^{300} (7i - 3)$
38. $\sum_{k=1}^{150} (11 + 2k)$

39. Find the sum of the first 1000 positive even integers.

40. What is the sum of the multiples of 3 between 3 and 999, inclusive?

Find the first three terms of each arithmetic series described.

41. $a_1 = 17, a_n = 197, S_n = 2247$

42. $a_1 = -13, a_n = 427, S_n = 18{,}423$

43. $n = 31, a_n = 78, S_n = 1023$

44. $n = 19, a_n = 103, S_n = 1102$

45. **AEROSPACE** On the Moon, a falling object falls just 2.65 feet in the first second after being dropped. Each second it falls 5.3 feet farther than in the previous second. How far would an object fall in the first ten seconds after being dropped?

CRITICAL THINKING **State whether each statement is *true* or *false*. Explain.**

46. Doubling each term in an arithmetic series will double the sum.

47. Doubling the number of terms in an arithmetic series, but keeping the first term and common difference the same, will double the sum.

48. **WRITING IN MATH** Answer the question that was posed at the beginning of the lesson.

How do arithmetic series apply to amphitheaters?

Include the following in your answer:

- explanations of what the sequence and the series that can be formed from the given numbers represent, and
- two ways to find the amphitheater capacity if it has ten rows of seats.

Standardized Test Practice

A B C D

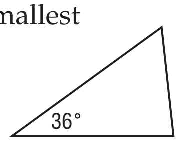
49. $18 + 22 + 26 + 30 + \ldots + 50 = ?$

(A) 146 (B) 272 (C) 306 (D) 340

50. The angles of a triangle form an arithmetic sequence. If the smallest angle measures 36°, what is the measure of the largest angle?

(A) 60° (B) 72°

(C) 84° (D) 144°



Graphing Calculator

Use a graphing calculator to find the sum of each arithmetic series.

51. $\sum_{n=21}^{75} (2n + 5)$

52. $\sum_{n=10}^{50} (3n - 1)$

53. $\sum_{n=20}^{60} (4n + 3)$

Maintain Your Skills

Mixed Review

Find the indicated term of each arithmetic sequence. *(Lesson 11-1)*

54. $a_1 = 46, d = 5, n = 14$

55. $a_1 = 12, d = -7, n = 22$

56. **RADIOACTIVITY** The decay of Radon-222 can be modeled by the equation $y = ae^{-0.1813t}$, where t is measured in days. What is the half-life of Radon-222? *(Lesson 10-6)*

Solve each equation by completing the square. *(Lesson 6-4)*

57. $x^2 + 9x + 20.25 = 0$

58. $9x^2 + 96x + 256 = 0$

59. $x^2 - 3x - 20 = 0$

Simplify. *(Lesson 5-6)*

60. $5\sqrt{3} - 4\sqrt{3}$

61. $\sqrt{26} \cdot \sqrt{39} \cdot \sqrt{14}$

62. $(\sqrt{10} - \sqrt{6})(\sqrt{5} + \sqrt{3})$

Getting Ready for the Next Lesson

PREREQUISITE SKILL **Evaluate the expression $a \cdot b^{n-1}$ for the given values of a, b, and n.** *(To review **evaluating expressions**, see Lesson 1-1.)*

63. $a = 1, b = 2, n = 5$

64. $a = 2, b = -3, n = 4$

65. $a = 18, b = \frac{1}{3}, n = 6$

11-3 Geometric Sequences

What You'll Learn

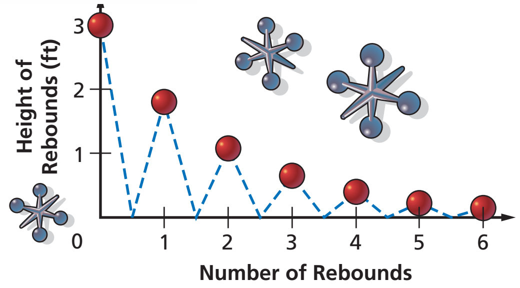
- Use geometric sequences.
- Find geometric means.

Vocabulary

- geometric sequence
- common ratio
- geometric means

How do geometric sequences apply to a bouncing ball?

If you have ever bounced a ball, you know that when you drop it, it never rebounds to the height from which you dropped it. Suppose a ball is dropped from a height of three feet, and each time it falls, it rebounds to 60% of the height from which it fell. The heights of the ball's rebounds form a sequence.



GEOMETRIC SEQUENCES The height of the first rebound of the ball is 3(0.6) or 1.8 feet. The height of the second rebound is 1.8(0.6) or 1.08 feet. The height of the third rebound is 1.08(0.6) or 0.648 feet. The sequence of heights, 1.8, 1.08, 0.648, …, is an example of a **geometric sequence**. A geometric sequence is a sequence in which each term after the first is found by multiplying the previous term by a constant r called the **common ratio**.

As with an arithmetic sequence, you can label the terms of a geometric sequence as a_1, a_2, a_3, and so on. The nth term is a_n and the previous term is a_{n-1}. So, $a_n = r(a_{n-1})$. Thus, $r = \frac{a_n}{a_{n-1}}$. That is, the common ratio can be found by dividing any term by its previous term.

Standardized Test Practice

Example 1 Find the Next Term

Multiple-Choice Test Item

Find the missing term in the geometric sequence: 8, 20, 50, 125, ___.

(A) 75 (B) 200 (C) 250 (D) 312.5

Read the Test Item

Since $\frac{20}{8} = 2.5$, $\frac{50}{20} = 2.5$, and $\frac{125}{50} = 2.5$, the sequence has a common ratio of 2.5.

Solve the Test Item

To find the missing term, multiply the last given term by 2.5: 125(2.5) = 312.5.

The answer is D.

The Princeton Review

Test-Taking Tip

Since the terms of this sequence are increasing, the missing term must be greater than 125. You can immediately eliminate 75 as a possible answer.

You have seen that each term of a geometric sequence can be expressed in terms of r and its previous term. It is also possible to develop a formula that expresses each term of a geometric sequence in terms of r and the first term a_1. Study the patterns shown in the table on the next page for the sequence 2, 6, 18, 54, … .

Sequence	numbers	2	6	18	54	…	
	symbols	a_1	a_2	a_3	a_4	…	a_n
Expressed in Terms of r and the Previous Term	numbers	2	2(3)	6(3)	18(3)	…	
	symbols	a_1	$a_1 \cdot r$	$a_2 \cdot r$	$a_3 \cdot r$	…	$a_{n-1} \cdot r$
Expressed in Terms of r and the First Term	numbers	2	2(3)	2(9)	2(27)	…	
		$2(3^0)$	$2(3^1)$	$2(3^2)$	$2(3^3)$	…	
	symbols	$a_1 \cdot r^0$	$a_1 \cdot r^1$	$a_1 \cdot r^2$	$a_1 \cdot r^3$	…	$a_1 \cdot r^{n-1}$

The three entries in the last column of the table all describe the nth term of a geometric sequence. This leads us to the following formula for finding the nth term of a geometric sequence.

Key Concept *nth Term of a Geometric Sequence*

The nth term a_n of a geometric sequence with first term a_1 and common ratio r is given by

$$a_n = a_1 \cdot r^{n-1},$$

where n is any positive integer.

Example 2 Find a Particular Term

Find the eighth term of a geometric sequence for which $a_1 = -3$ and $r = -2$.

$a_n = a_1 \cdot r^{n-1}$ Formula for nth term

$a_8 = (-3) \cdot (-2)^{8-1}$ $n = 8, a_1 = -3, r = -2$

$a_8 = (-3) \cdot (-128)$ $(-2)^7 = -128$

$a_8 = 384$ Multiply.

The eighth term is 384.

Example 3 Write an Equation for the nth Term

Write an equation for the nth term of the geometric sequence 3, 12, 48, 192, … .

In this sequence, $a_1 = 3$ and $r = 4$. Use the nth term formula to write an equation.

$a_n = a_1 \cdot r^{n-1}$ Formula for nth term

$a_n = 3 \cdot 4^{n-1}$ $a_1 = 3, r = 4$

An equation is $a_n = 3 \cdot 4^{n-1}$.

You can also use the formula for the nth term if you know the common ratio and one term of a geometric sequence, but not the first term.

Example 4 Find a Term Given the Fourth Term and the Ratio

Find the tenth term of a geometric sequence for which $a_4 = 108$ and $r = 3$.

First, find the value of a_1.

$a_n = a_1 \cdot r^{n-1}$ Formula for nth term

$a_4 = a_1 \cdot 3^{4-1}$ $n = 4, r = 3$

$108 = 27a_1$ $a_4 = 108$

$4 = a_1$ Divide each side by 27.

Now find a_{10}.

$a_n = a_1 \cdot r^{n-1}$ Formula for nth term

$a_{10} = 4 \cdot 3^{10-1}$ $n = 10, a_1 = 4, r = 3$

$a_{10} = 78{,}732$ Use a calculator.

The tenth term is 78,732.

GEOMETRIC MEANS In Lesson 11-1, you learned that missing terms between two nonsuccessive terms in an arithmetic sequence are called *arithmetic means*. Similarly, the missing terms(s) between two nonsuccessive terms of a geometric sequence are called **geometric means**. For example, 6, 18, and 54 are three geometric means between 2 and 162 in the sequence 2, 6, 18, 54, 162, … . You can use the common ratio to find the geometric means in a given sequence.

Example 5 Find Geometric Means

Find three geometric means between 2.25 and 576.

Use the *n*th term formula to find the value of r. In the sequence 2.25, __?__, __?__, __?__, 576, a_1 is 2.25 and a_5 is 576.

$a_n = a_1 \cdot r^{n-1}$	Formula for *n*th term
$a_5 = 2.25 \cdot r^{5-1}$	$n = 5, a_1 = 2.25$
$576 = 2.25r^4$	$a_5 = 576$
$256 = r^4$	Divide each side by 2.25.
$\pm 4 = r$	Take the fourth root of each side.

There are two possible common ratios, so there are two possible sets of geometric means. Use each value of r to find three geometric means.

$r = 4$	$r = -4$
$a_2 = 2.25(4)$ or 9	$a_2 = 2.25(-4)$ or -9
$a_3 = 9(4)$ or 36	$a_3 = -9(-4)$ or 36
$a_4 = 36(4)$ or 144	$a_4 = 36(-4)$ or -144

The geometric means are 9, 36, and 144, or -9, 36, and -144.

Study Tip

Alternate Method

You may prefer this method. The three means will be $2.25r$, $2.25r^2$, and $2.25r^3$. Then the common ratio is $r = \frac{576}{2.25r^3}$ or $r^4 = \frac{576}{2.25}$. Thus, $r = 4$.

Check for Understanding

Concept Check

1. **Decide** whether each sequence is *arithmetic* or *geometric*. Explain.
   a. 1, −2, 4, −8, … **b.** 1, −2, −5, −8, …

2. **OPEN ENDED** Write a geometric sequence with a common ratio of $\frac{2}{3}$.

3. **FIND THE ERROR** Marika and Lori are finding the seventh term of the geometric sequence 9, 3, 1, … .

Marika	Lori
$r = \frac{3}{9}$ or $\frac{1}{3}$	$r = \frac{9}{3}$ or 3
$a_7 = 9\left(\frac{1}{3}\right)^{7-1}$	$a_7 = 9 \cdot 3^{7-1}$
$= \frac{1}{81}$	$= 6561$

Who is correct? Explain your reasoning.

Guided Practice

Find the next two terms of each geometric sequence.

4. 20, 30, 45, … 5. $-\frac{1}{4}, \frac{1}{2}, -1, \ldots$

6. Find the first five terms of the geometric sequence for which $a_1 = -2$ and $r = 3$.

7. Find a_9 for the geometric sequence 60, 30, 15, … .

Find the indicated term of each geometric sequence.

8. $a_1 = 7, r = 2, n = 4$

9. $a_3 = 32, r = -0.5, n = 6$

10. Write an equation for the nth term of the geometric sequence 4, 8, 16, … .

11. Find two geometric means between 1 and 27.

Standardized Test Practice

12. Find the missing term in the geometric sequence: $\frac{9}{4}, \frac{3}{4}, \frac{1}{4}, \frac{1}{12}$, ____.

Ⓐ $\frac{1}{36}$ Ⓑ $\frac{1}{20}$ Ⓒ $\frac{1}{6}$ Ⓓ $\frac{1}{3}$

Practice and Apply

Homework Help

For Exercises	See Examples
13–24	1
25–30, 33–38, 47, 48	2
31, 32	4
39–42	3
43–46	5

Extra Practice

See page 852.

Find the next two terms of each geometric sequence.

13. 405, 135, 45, …

14. 81, 108, 144, …

15. 16, 24, 36, …

16. 162, 108, 72, …

17. $\frac{5}{2}, \frac{5}{3}, \frac{10}{9}, \ldots$

18. $\frac{1}{3}, \frac{5}{6}, \frac{25}{12}, \ldots$

19. 1.25, −1.5, 1.8, …

20. 1.4, −3.5, 8.75, …

Find the first five terms of each geometric sequence described.

21. $a_1 = 2, r = -3$

22. $a_1 = 1, r = 4$

23. $a_1 = 243, r = \frac{1}{3}$

24. $a_1 = 576, r = -\frac{1}{2}$

25. Find a_7 if $a_n = 12\left(\frac{1}{2}\right)^{n-1}$.

26. If $a_n = \frac{1}{3} \cdot 6^{n-1}$, what is a_6?

Find the indicated term of each geometric sequence.

27. $a_1 = \frac{1}{3}, r = 3, n = 8$

28. $a_1 = \frac{1}{64}, r = 4, n = 9$

29. $a_1 = 16{,}807, r = \frac{3}{7}, n = 6$

30. $a_1 = 4096, r = \frac{1}{4}, n = 8$

31. $a_4 = 16, r = 0.5, n = 8$

32. $a_6 = 3, r = 2, n = 12$

33. a_9 for $\frac{1}{5}, 1, 5, \ldots$

34. a_7 for $\frac{1}{32}, \frac{1}{16}, \frac{1}{8}, \ldots$

35. a_8 for 4, −12, 36, …

36. a_6 for 540, 90, 15, …

More About. . .

Art

The largest ever ice construction was an ice palace built for a carnival in St. Paul, Minnesota, in 1992. It contained 10.8 million pounds of ice.

Source: *The Guinness Book of Records*

37. **ART** A one-ton ice sculpture is melting so that it loses one-fifth of its weight per hour. How much of the sculpture will be left after five hours? Write the answer in pounds.

38. **SALARIES** Geraldo's current salary is $40,000 per year. His annual pay raise is always a percent of his salary at the time. What would his salary be if he got four consecutive 4% increases?

Write an equation for the *n*th term of each geometric sequence.

39. 36, 12, 4, …

40. 64, 16, 4, …

41. −2, 10, −50, …

42. 4, −12, 36, …

Find the geometric means in each sequence.

43. 9, __?__, __?__, __?__, 144

44. 4, __?__, __?__, __?__, 324

45. 32, __?__, __?__, __?__, __?__, 1

46. 3, __?__, __?__, __?__, __?__, 96

MEDICINE **For Exercises 47 and 48, use the following information.**
Iodine-131 is a radioactive element used to study the thyroid gland.

47. **RESEARCH** Use the Internet or other resource to find the *half-life* of Iodine-131, rounded to the nearest day. This is the amount of time it takes for half of a sample of Iodine-131 to decay into another element.

48. How much of an 80-milligram sample of Iodine-131 would be left after 32 days?

CRITICAL THINKING **Determine whether each statement is *true* or *false*. If true, explain. If false, provide a counterexample.**

49. Every sequence is either arithmetic or geometric.

50. There is no sequence that is both arithmetic and geometric.

51. **WRITING IN MATH** Answer the question that was posed at the beginning of the lesson.

How do geometric sequences apply to a bouncing ball?

Include the following in your answer:
- the first five terms of the sequence of heights from which the ball falls, and
- any similarities or differences in the sequences for the heights the ball rebounds and the heights from which the ball falls.

Standardized Test Practice

52. Find the missing term in the geometric sequence: $-5, 10, -20, 40,$ ___.

Ⓐ -80 Ⓑ -35 Ⓒ 80 Ⓓ 100

53. What is the tenth term in the geometric sequence: 144, 72, 36, 18, …?

Ⓐ 0 Ⓑ $\frac{9}{64}$ Ⓒ $\frac{9}{32}$ Ⓓ $\frac{9}{16}$

Maintain Your Skills

Mixed Review

Find S_n for each arithmetic series described. *(Lesson 11-2)*

54. $a_1 = 11, a_n = 44, n = 23$

55. $a_1 = -5, d = 3, n = 14$

Find the arithmetic means in each sequence. *(Lesson 11-1)*

56. 15, __?__, __?__, 27

57. -8, __?__, __?__, __?__, -24

58. **GEOMETRY** Find the perimeter of a triangle with vertices at $(2, 4)$, $(-1, 3)$ and $(1, -3)$. *(Lesson 8-1)*

Getting Ready for the Next Lesson

PREREQUISITE SKILL **Evaluate each expression.** *(To review **expressions**, see Lesson 1-1.)*

59. $\dfrac{1 - 2^7}{1 - 2}$

60. $\dfrac{1 - \left(\frac{1}{2}\right)^6}{1 - \left(\frac{1}{2}\right)}$

61. $\dfrac{1 - \left(-\frac{1}{3}\right)^5}{1 - \left(-\frac{1}{3}\right)}$

Practice Quiz 1

Lessons 11-1 through 11-3

Find the indicated term of each arithmetic sequence. *(Lesson 11-1)*

1. $a_1 = 7, d = 3, n = 14$

2. $a_1 = 2, d = \frac{1}{2}, n = 8$

Find the sum of each arithmetic series described. *(Lesson 11-2)*

3. $a_1 = 5, a_n = 29, n = 11$

4. $6 + 12 + 18 + \ldots + 96$

5. Find a_7 for the geometric sequence $729, -243, 81, \ldots$. *(Lesson 11-3)*

Limits

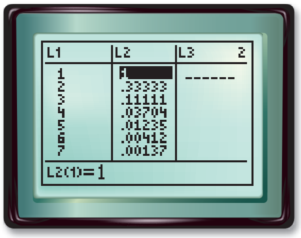
You may have noticed that in some geometric sequences, the later the term in the sequence, the closer the value is to 0. Another way to describe this is that as n increases, a_n approaches 0. The value that the terms of a sequence approach, in this case 0, is called the **limit** of the sequence. Other types of infinite sequences may also have limits. If the terms of a sequence do not approach a unique value, we say that the limit of the sequence does not exist.

Find the limit of the geometric sequence $1, \frac{1}{3}, \frac{1}{9}, \ldots$.

Step 1 *Enter the sequence.*

- The formula for this sequence is $a_n = \left(\frac{1}{3}\right)^{n-1}$.
- Position the cursor on L1 in the STAT EDIT Edit … screen and enter the formula seq(N,N,1,10,1). This generates the values 1, 2, …, 10 of the index N.
- Position the cursor on L2 and enter the formula seq((1/3)^(N-1),N,1,10,1). This generates the first ten terms of the sequence.

KEYSTROKES: *Review sequences in the Graphing Calculator Investigation on page 585.*

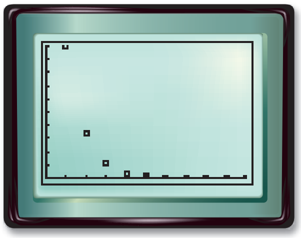


Notice that as n increases, the terms of the given sequence get closer and closer to 0. If you scroll down, you can see that for $n \geq 8$ the terms are so close to 0 that the calculator expresses them in scientific notation. This suggests that the limit of the sequence is 0.

Step 2 *Graph the sequence.*

- Use a STAT PLOT to graph the sequence. Use L1 as the Xlist and L2 as the Ylist.

KEYSTROKES: *Review STAT PLOTs on page 87.*



[0, 10] scl: 1 by [0, 1] scl: 0.1

The graph also shows that, as n increases, the terms approach 0. In fact, for $n \geq 6$, the marks appear to lie on the horizontal axis. This strongly suggests that the limit of the sequence is 0.

Exercises

Use a graphing calculator to find the limit, if it exists, of each sequence.

1. $a_n = \left(\frac{1}{2}\right)^n$

2. $a_n = \left(-\frac{1}{2}\right)^n$

3. $a_n = 4^n$

4. $a_n = \frac{1}{n^2}$

5. $a_n = \frac{2^n}{2^n + 1}$

6. $a_n = \frac{n^2}{n + 1}$

www.algebra2.com/other_calculator_keystrokes

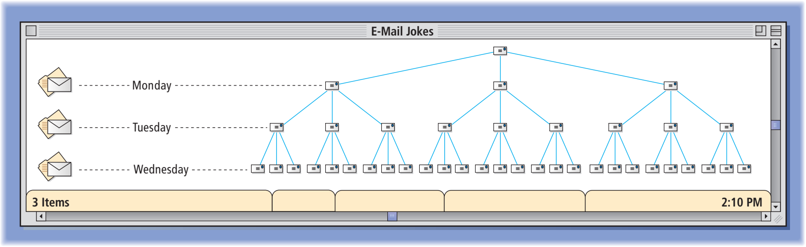
11-4 Geometric Series

What You'll Learn

- Find sums of geometric series.
- Find specific terms of geometric series.

How is e-mailing a joke like a geometric series?

Suppose you e-mail a joke to three friends on Monday. Each of those friends sends the joke on to three of their friends on Tuesday. Each person who receives the joke on Tuesday sends it to three more people on Wednesday, and so on.



Vocabulary

- geometric series

GEOMETRIC SERIES Notice that every day, the number of people who read your joke is three times the number that read it the day before. By Sunday, the number of people, including yourself, who have read the joke is 1 + 3 + 9 + 27 + 81 + 243 + 729 + 2187 or 3280!

The numbers 1, 3, 9, 27, 81, 243, 729, and 2187 form a geometric sequence in which $a_1 = 1$ and $r = 3$. Since 1, 3, 9, 27, 81, 243, 729, 2187 is a geometric sequence, 1 + 3 + 9 + 27 + 81 + 243 + 729 + 2187 is called a **geometric series**. Below are some more examples of geometric sequences and their corresponding geometric series.

Geometric Sequences	Geometric Series
1, 2, 4, 8, 16	1 + 2 + 4 + 8 + 16
4, −12, 36	4 + (−12) + 36
$5, 1, \frac{1}{5}, \frac{1}{25}$	$5 + 1 + \frac{1}{5} + \frac{1}{25}$

To develop a formula for the sum of a geometric series, consider the series given in the e-mail situation above.

$$\begin{aligned} S_8 &= 1 + 3 + 9 + 27 + 81 + 243 + 729 + 2187 \\ (-)\ 3S_8 &= \phantom{1 + {}} 3 + 9 + 27 + 81 + 243 + 729 + 2187 + 6561 \\ \hline (1-3)S_8 &= 1 + 0 + 0 + 0 + 0 + 0 + 0 + 0 - 6561 \end{aligned}$$

$$S_8 = \frac{1 - 6561}{1 - 3} \text{ or } 3280$$

first term in series

last term in series multiplied by common ratio; in this case, a_9

common ratio

Study Tip

Terms of Geometric Sequences

Remember that a_9 can also be written as a_1r^8.

The expression for S_8 can be written as $S_8 = \frac{a_1 - a_1r^8}{1 - r}$. A rational expression like this can be used to find the sum of any geometric series.

Key Concept — *Sum of a Geometric Series*

The sum S_n of the first n terms of a geometric series is given by

$S_n = \dfrac{a_1 - a_1 r^n}{1 - r}$ or $S_n = \dfrac{a_1(1 - r^n)}{1 - r}$, where $r \neq 1$.

You cannot use the formula for the sum with a geometric series for which $r = 1$ because division by 0 would result. In a geometric series with $r = 1$, the terms are constant. For example, $4 + 4 + 4 + \ldots + 4$ is such a series. In general, the sum of n terms of a geometric series with $r = 1$ is $n \cdot a^1$.

Example 1 — Find the Sum of the First n Terms

GENEALOGY **In the book *Roots*, author Alex Haley traced his family history back many generations to the time one of his ancestors was brought to America from Africa. If you could trace your family back for 15 generations, starting with your parents, how many ancestors would there be?**

Counting your two parents, four grandparents, eight great-grandparents, and so on gives you a geometric series with $a_1 = 2$, $r = 2$, and $n = 15$.

$$S_n = \frac{a_1(1 - r^n)}{1 - r} \quad \text{Sum formula}$$

$$S_{15} = \frac{2(1 - 2^{15})}{1 - 2} \quad n = 15,\ a_1 = 2,\ r = 2$$

$$S_{15} = 65{,}534 \quad \text{Use a calculator.}$$

Going back 15 generations, you have 65,534 ancestors.

More About. . .

Genealogy

When he died in 1992, Samuel Must of Fryburg, Pennsylvania, had a record 824 living descendants.

Source: *The Guinness Book of Records*

As with arithmetic series, you can use sigma notation to represent geometric series.

Example 2 — Evaluate a Sum Written in Sigma Notation

Evaluate $\sum_{n=1}^{6} 5 \cdot 2^{n-1}$.

Method 1

Find the terms by replacing n with 1, 2, 3, 4, 5, and 6. Then add.

$$\begin{aligned}\sum_{n=1}^{6} 5 \cdot 2^{n-1} &= 5(2^{1-1}) + 5(2^{2-1}) + 5(2^{3-1}) + 5(2^{4-1}) + 5(2^{5-1}) + 5(2^{6-1})\\ &= 5(1) + 5(2) + 5(4) + 5(8) + 5(16) + 5(32)\\ &= 5 + 10 + 20 + 40 + 80 + 160\\ &= 315\end{aligned}$$

The sum of the series is 315.

Method 2

Since the sum is a geometric series, you can use the formula

$$S_n = \frac{a_1(1 - r^n)}{1 - r}.$$

$$S_6 = \frac{5(1 - 2^6)}{1 - 2} \quad n = 6,\ a_1 = 5,\ r = 2$$

$$S_6 = \frac{5(-63)}{-1} \quad 2^6 = 64$$

$$S_6 = 315 \quad \text{Simplify.}$$

How can you find the sum of a geometric series if you know the first and last terms and the common ratio, but not the number of terms? Remember the formula for the nth term of a geometric sequence or series, $a_n = a_1 \cdot r^{n-1}$. You can use this formula to find an expression involving r^n.

$$a_n = a_1 \cdot r^{n-1} \quad \text{Formula for } n\text{th term}$$

$$a_n \cdot r = a_1 \cdot r^{n-1} \cdot r \quad \text{Multiply each side by } r.$$

$$a_n \cdot r = a_1 \cdot r^n \quad r^{n-1} \cdot r^1 = r^{n-1+1} \text{ or } r^n$$

Now substitute $a_n \cdot r$ for $a_1 \cdot r^n$ in the formula for the sum of a geometric series. The result is $S_n = \frac{a_1 - a_n r}{1 - r}$.

Example 3 Use the Alternate Formula for a Sum

Find the sum of a geometric series for which $a_1 = 15{,}625$, $a_n = -5$, and $r = -\frac{1}{5}$.

Since you do not know the value of n, use the formula derived above.

$S_n = \frac{a_1 - a_n r}{1 - r}$ — Alternate sum formula

$= \frac{15{,}625 - (-5)\left(-\frac{1}{5}\right)}{1 - \left(-\frac{1}{5}\right)}$ — $a_1 = 15{,}625$, $a_n = -5$, $r = -\frac{1}{5}$

$= \frac{15{,}624}{\frac{6}{5}}$ or 13,020 — Simplify.

SPECIFIC TERMS You can use the formula for the sum of a geometric series to help find a particular term of the series.

Example 4 Find the First Term of a Series

Find a_1 in a geometric series for which $S_8 = 39{,}360$ and $r = 3$.

$S_n = \frac{a_1(1 - r^n)}{1 - r}$ — Sum formula

$39{,}360 = \frac{a_1(1 - 3^8)}{1 - 3}$ — $S_8 = 39{,}360$; $r = 3$; $n = 8$

$39{,}360 = \frac{-6560a_1}{-2}$ — Subtract.

$39{,}360 = 3280a_1$ — Divide.

$12 = a_1$ — Divide each side by 3280.

The first term of the series is 12.

Check for Understanding

Concept Check

1. **OPEN ENDED** Write a geometric series for which $r = \frac{1}{2}$ and $n = 4$.
2. **Explain**, using geometric series, why the polynomial $1 + x + x^2 + x^3$ can be written as $\frac{x^4 - 1}{x - 1}$, assuming $x \neq 1$.
3. **Explain** how to write the series 2 + 12 + 72 + 432 + 2592 using sigma notation.

Guided Practice

Find S_n for each geometric series described.

4. $a_1 = 12$, $a_5 = 972$, $r = -3$
5. $a_1 = 3$, $a_n = 46{,}875$, $r = -5$
6. $a_1 = 5$, $r = 2$, $n = 14$
7. $a_1 = 243$, $r = -\frac{2}{3}$, $n = 5$

Find the sum of each geometric series.

8. 54 + 36 + 24 + 16 + … to 6 terms
9. 3 − 6 + 12 − … to 7 terms
10. $\sum_{n=1}^{5} \frac{1}{4} \cdot 2^{n-1}$
11. $\sum_{n=1}^{7} 81\left(\frac{1}{3}\right)^{n-1}$

Find the indicated term for each geometric series described.

12. $S_n = \frac{381}{64}, r = \frac{1}{2}, n = 7; a_1$

13. $S_n = 33, a_n = 48, r = -2; a_1$

Application

14. WEATHER Heavy rain caused a river to rise. The river rose three inches the first day, and each additional day it rose twice as much as the previous day. How much did the river rise in five days?

Practice and Apply

Homework Help

For Exercises	See Examples
15–34, 47	1, 3
35–40	2
41–46	4

Extra Practice

See page 852.

Find S_n for each geometric series described.

15. $a_1 = 2, a_6 = 486, r = 3$

16. $a_1 = 3, a_8 = 384, r = 2$

17. $a_1 = 1296, a_n = 1, r = -\frac{1}{6}$

18. $a_1 = 343, a_n = -1, r = -\frac{1}{7}$

19. $a_1 = 4, r = -3, n = 5$

20. $a_1 = 5, r = 3, n = 12$

21. $a_1 = 2401, r = -\frac{1}{7}, n = 5$

22. $a_1 = 625, r = \frac{3}{5}, n = 5$

23. $a_1 = 162, r = \frac{1}{3}, n = 6$

24. $a_1 = 80, r = -\frac{1}{2}, n = 7$

25. $a_1 = 625, r = 0.4, n = 8$

26. $a_1 = 4, r = 0.5, n = 8$

27. $a_2 = -36, a_5 = 972, n = 7$

28. $a_3 = -36, a_6 = -972, n = 10$

29. HEALTH Contagious diseases can spread very quickly. Suppose five people are ill during the first week of an epidemic and that each person who is ill spreads the disease to four people by the end of the next week. By the end of the tenth week of the epidemic, how many people have been affected by the illness?

30. LEGENDS There is a legend of a king who wanted to reward a boy for a good deed. The king gave the boy a choice. He could have \$1,000,000 at once, or he could be rewarded daily for a 30-day month, with one penny on the first day, two pennies on the second day, and so on, receiving twice as many pennies each day as the previous day. How much would the second option be worth?

More About. . .

Legends

Some of the best-known legends involving a king are the Arthurian legends. According to legend, King Arthur reigned over Britain before the Saxon conquest. Camelot was the most famous castle in the medieval legends of King Arthur.

Find the sum of each geometric series.

31. $4096 - 512 + 64 - \ldots$ to 5 terms

32. $7 + 21 + 63 + \ldots$ to 10 terms

33. $\frac{1}{16} + \frac{1}{4} + 1 + \ldots$ to 7 terms

34. $\frac{1}{9} - \frac{1}{3} + 1 - \ldots$ to 6 terms

35. $\sum_{n=1}^{9} 5 \cdot 2^{n-1}$

36. $\sum_{n=1}^{6} 2(-3)^{n-1}$

37. $\sum_{n=1}^{7} 144\left(-\frac{1}{2}\right)^{n-1}$

38. $\sum_{n=1}^{8} 64\left(\frac{3}{4}\right)^{n-1}$

39. $\sum_{n=1}^{20} 3 \cdot 2^{n-1}$

40. $\sum_{n=1}^{16} 4 \cdot 3^{n-1}$

Find the indicated term for each geometric series described.

41. $S_n = 165, a_n = 48, r = -\frac{2}{3}; a_1$

42. $S_n = 688, a_n = 16, r = -\frac{1}{2}; a_1$

43. $S_n = -364, r = -3, n = 6; a_1$

44. $S_n = 1530, r = 2, n = 8; a_1$

45. $S_n = 315, r = 0.5, n = 6; a_2$

46. $S_n = 249.92, r = 0.2, n = 5, a_3$

47. LANDSCAPING Rob is helping his dad install a fence. He is using a sledgehammer to drive the pointed fence posts into the ground. On his first swing, he drives a post five inches into the ground. Since the soil is denser the deeper he drives, on each swing after the first, he can only drive the post 30% as far into the ground as he did on the previous swing. How far has he driven the post into the ground after five swings?

48. **CRITICAL THINKING** If a_1 and r are integers, explain why the value of $\frac{a_1 - a_1r^n}{1 - r}$ must also be an integer.

49. ***WRITING IN MATH*** Answer the question that was posed at the beginning of the lesson.

How is e-mailing a joke like a geometric series?

Include the following in your answer:

- how the related geometric series would change if each person e-mailed the joke on to four people instead of three, and
- how the situation could be changed to make it better to use a formula than to add terms.

Standardized Test Practice

50. The first term of a geometric series is −1, and the common ratio is −3. How many terms are in the series if its sum is 182?

(A) 6 (B) 7 (C) 8 (D) 9

51. What is the first term in a geometric series with ten terms, a common ratio of 0.5, and a sum of 511.5?

(A) 64 (B) 128 (C) 256 (D) 512

Graphing Calculator

Use a graphing calculator to find the sum of each geometric series.

52. $\sum_{n=1}^{20} 3(-2)^{n-1}$ **53.** $\sum_{n=1}^{15} 2\left(\frac{1}{2}\right)^{n-1}$ **54.** $\sum_{n=1}^{10} 5(0.2)^{n-1}$

Maintain Your Skills

Mixed Review

Find the geometric means in each sequence. *(Lesson 11-3)*

55. $\frac{1}{24}$, __?__, __?__, __?__, 54 **56.** −2, __?__, __?__, __?__, __?__, $-\frac{243}{16}$

Find the sum of each arithmetic series. *(Lesson 11-2)*

57. 50 + 44 + 38 + … + 8 **58.** $\sum_{n=1}^{12} (2n + 3)$

ENTERTAINMENT **For Exercises 59–61, use the table that shows the number of drive-in movie screens in the United States for 1995–2000.** *(Lesson 2-5)*

Year	1995	1996	1997	1998	1999	2000
Screens	848	826	815	750	737	637

Source: National Association of Theatre Owners

59. Draw a scatter plot, in which x is the number of years since 1995.

60. Find a prediction equation.

61. Predict the number of screens in 2010.

Online Research **Data Update** For the latest statistics on the movie industry, visit: www.algebra2.com/data_update

Getting Ready for the Next Lesson

PREREQUISITE SKILL **Evaluate $\frac{a}{1 - b}$ for the given values of a and b.**

*(To review **evaluating expressions**, see Lesson 1-1.)*

62. $a = 1, b = \frac{1}{2}$ **63.** $a = 3, b = -\frac{1}{2}$ **64.** $a = \frac{1}{3}, b = -\frac{1}{3}$

65. $a = \frac{1}{2}, b = \frac{1}{4}$ **66.** $a = -1, b = 0.5$ **67.** $a = 0.9, b = -0.5$

11-5 Infinite Geometric Series

What You'll Learn

- Find the sum of an infinite geometric series.
- Write repeating decimals as fractions.

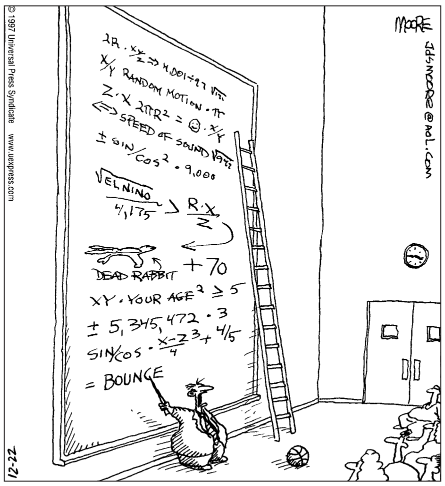
Vocabulary

- infinite geometric series
- partial sum

How does an infinite geometric series apply to a bouncing ball?

Refer to the beginning of Lesson 11-3. Suppose you wrote a geometric series to find the sum of the heights of the rebounds of the ball. The series would have no last term because theoretically there is no last bounce of the ball. For every rebound of the ball, there is another rebound, 60% as high. Such a geometric series is called an **infinite geometric series**.

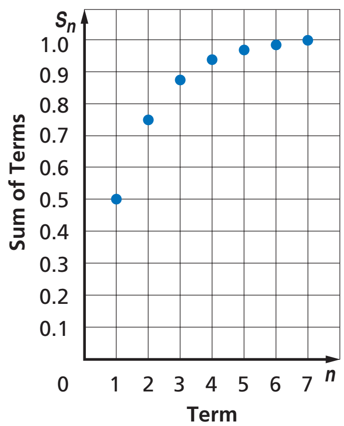
In the Bleachers By Steve Moore



"And that, ladies and gentlemen, is the way the ball bounces."

INFINITE GEOMETRIC SERIES Consider the infinite geometric series $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$. You have already learned how to find the sum S_n of the first n terms of a geometric series. For an infinite series, S_n is called a **partial sum** of the series. The table and graph show some values of S_n.

n	S_n
1	$\frac{1}{2}$ or 0.5
2	$\frac{3}{4}$ or 0.75
3	$\frac{7}{8}$ or 0.875
4	$\frac{15}{16}$ or 0.9375
5	$\frac{31}{32}$ or 0.96875
6	$\frac{63}{64}$ or 0.984375
7	$\frac{127}{128}$ or 0.9921875



Study Tip

Absolute Value

Recall that $|r| < 1$ *means* $-1 < r < 1$.

Notice that as n increases, the partial sums level off and approach a limit of 1. This leveling-off behavior is characteristic of infinite geometric series for which $|r| < 1$.

Let's look at the formula for the sum of a finite geometric series and use it to find a formula for the sum of an infinite geometric series.

$$S_n = \frac{a_1 - a_1 r^n}{1 - r} \quad \text{Sum of first } n \text{ terms}$$

$$= \frac{a_1}{1 - r} - \frac{a_1 r^n}{1 - r} \quad \text{Write the fraction as a difference of fractions.}$$

If $-1 < r < 1$, the value of r^n will approach 0 as n increases. Therefore, the partial sums of an infinite geometric series will approach $\frac{a_1}{1 - r} - \frac{a_1(0)}{1 - r}$ or $\frac{a_1}{1 - r}$. This expression gives the sum of an infinite geometric series.

Key Concept — *Sum of an Infinite Geometric Series*

The sum S of an infinite geometric series with $-1 < r < 1$ is given by

$$S = \frac{a_1}{1 - r}.$$

Study Tip

Formula for Sum if $-1 < r < 1$

To convince yourself of this formula, make a table of the first ten partial sums of the geometric series with $r = \frac{1}{2}$ and $a_1 = 100$.

Term Number	Term	Partial Sum
1	100	100
2	50	150
3	25	175
⋮	⋮	⋮
10		

Complete the table and compare the sum that the series is approaching to that obtained by using the formula.

An infinite geometric series for which $|r| \geq 1$ does not have a sum. Consider the series $1 + 3 + 9 + 27 + 81 + \ldots$. In this series, $a_1 = 1$ and $r = 3$. The table shows some of the partial sums of this series. As n increases, S_n rapidly increases and has no limit. That is, the partial sums do not approach a particular value.

n	S_n
5	121
10	29,524
15	7,174,453
20	1,743,392,200

Example 1 Sum of an Infinite Geometric Series

Find the sum of each infinite geometric series, if it exists.

a. $\frac{1}{2} + \frac{3}{8} + \frac{9}{32} + \ldots$

First, find the value of r to determine if the sum exists.

$a_1 = \frac{1}{2}$ and $a_2 = \frac{3}{8}$, so $r = \frac{\frac{3}{8}}{\frac{1}{2}}$ or $\frac{3}{4}$. Since $\left|\frac{3}{4}\right| < 1$, the sum exists.

Now use the formula for the sum of an infinite geometric series.

$$S = \frac{a_1}{1 - r} \quad \text{Sum formula}$$

$$= \frac{\frac{1}{2}}{1 - \frac{3}{4}} \quad a_1 = \frac{1}{2}, r + \frac{3}{4}$$

$$= \frac{\frac{1}{2}}{\frac{1}{4}} \text{ or } 2 \quad \text{Simplify.}$$

The sum of the series is 2.

b. $\mathbf{1 - 2 + 4 - 8 + \ldots}$

$a_1 = 1$ and $a_2 = -2$, so $r = \frac{-2}{1}$ or -2. Since $|-2| \geq 1$, the sum does not exist.

In Lessons 11-2 and 11-4, we used sigma notation to represent finite series. You can also use sigma notation to represent infinite series. An *infinity symbol* ∞ is placed above the Σ to indicate that a series is infinite.

Example 2 Infinite Series in Sigma Notation

Evaluate $\sum_{n=1}^{\infty} 24\left(-\frac{1}{5}\right)^{n-1}$.

In this infinite geometric series, $a_1 = 24$ and $r = -\frac{1}{5}$.

$S = \frac{a_1}{1 - r}$ Sum formula

$= \frac{24}{1 - \left(-\frac{1}{5}\right)}$ $a_1 = 24, r = -\frac{1}{5}$

$= \frac{24}{\frac{6}{5}}$ or 20 Simplify.

Thus, $\sum_{n=1}^{\infty} 24\left(-\frac{1}{5}\right)^{n-1} = 20$.

REPEATING DECIMALS The formula for the sum of an infinite geometric series can be used to write a repeating decimal as a fraction. Remember that decimals with bar notation such as $0.\overline{2}$ and $0.\overline{47}$ represent 0.222222… and 0.474747…, respectively. Each of these expressions can be written as an infinite geometric series.

Example 3 Write a Repeating Decimal as a Fraction

Write $0.\overline{39}$ as a fraction.

Method 1

Write the repeating decimal as a sum.

$0.\overline{39} = 0.393939\ldots$

$= 0.39 + 0.0039 + 0.000039 + \ldots$

$= \frac{39}{100} + \frac{39}{10{,}000} + \frac{39}{1{,}000{,}000} + \ldots$

In this series, $a_1 = \frac{39}{100}$ and $r = \frac{1}{100}$.

$S = \frac{a_1}{1 - r}$ Sum formula

$= \frac{\frac{39}{100}}{1 - \frac{1}{100}}$ $a_1 = \frac{39}{100}, r = \frac{1}{100}$

$= \frac{\frac{39}{100}}{\frac{99}{100}}$ Subtract.

$= \frac{39}{99}$ or $\frac{13}{33}$ Simplify.

Method 2

$S = 0.\overline{39}$ Label the given decimal.

$S = 0.393939\ldots$ Repeating decimal

$100S = 39.393939\ldots$ Multiply each side by 100.

$99S = 39$ Subtract the second equation from the third.

$S = \frac{39}{99}$ or $\frac{13}{33}$ Divide each side by 99.

Thus, $0.\overline{39} = \frac{13}{33}$.

www.algebra2.com/extra_examples

Check for Understanding

Concept Check

1. **OPEN ENDED** Write the series $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$ using sigma notation.

2. **Explain** why $0.999999\ldots = 1$.

3. **FIND THE ERROR** Miguel and Beth are discussing the series $-\frac{1}{3} + \frac{4}{9} - \frac{16}{27} + \ldots$. Miguel says that the sum of the series is $-\frac{1}{7}$. Beth says that the series does not have a sum. Who is correct? Explain your reasoning.

Miguel

$$S = \frac{-\frac{1}{3}}{1 - \left(-\frac{4}{3}\right)}$$
$$= -\frac{1}{7}$$

Guided Practice

Find the sum of each infinite geometric series, if it exists.

4. $a_1 = 36, r = \frac{2}{3}$

5. $a_1 = 18, r = -1.5$

6. $16 + 24 + 36 + \ldots$

7. $\frac{1}{4} + \frac{1}{6} + \frac{2}{18} + \ldots$

8. $6 - 2.4 + 0.96 - \ldots$

9. $\sum_{n=1}^{\infty} 40\left(\frac{3}{5}\right)^{n-1}$

Write each repeating decimal as a fraction.

10. $0.\overline{5}$

11. $0.\overline{73}$

12. $0.\overline{175}$

Application

13. **CLOCKS** Jasmine's old grandfather clock is broken. When she tries to set the pendulum in motion by holding it against the side of the clock and letting it go, it first swings 24 centimeters to the other side, then 18 centimeters back, then 13.5 centimeters, and so on. What is the total distance that the pendulum swings?

Practice and Apply

Homework Help

For Exercises	See Examples
14–27, 32–39	1
28–31	2
40–47	3

Extra Practice

See page 852.

Find the sum of each infinite geometric series, if it exists.

14. $a_1 = 4, r = \frac{5}{7}$

15. $a_1 = 14, r = \frac{7}{3}$

16. $a_1 = 12, r = -0.6$

17. $a_1 = 18, r = 0.6$

18. $16 + 12 + 9 + \ldots$

19. $-8 - 4 - 2 - \ldots$

20. $12 - 18 + 24 - \ldots$

21. $18 - 12 + 8 - \ldots$

22. $1 + \frac{2}{3} + \frac{4}{9} + \ldots$

23. $\frac{5}{3} + \frac{25}{3} + \frac{125}{3} + \ldots$

24. $\frac{5}{3} - \frac{10}{9} + \frac{20}{27} - \ldots$

25. $\frac{3}{2} - \frac{3}{4} + \frac{3}{8} - \ldots$

26. $3 + 1.8 + 1.08 + \ldots$

27. $1 - 0.5 + 0.25 - \ldots$

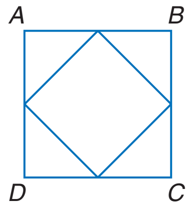
28. $\sum_{n=1}^{\infty} 48\left(\frac{2}{3}\right)^{n-1}$

29. $\sum_{n=1}^{\infty} \left(\frac{3}{8}\right)\left(\frac{3}{4}\right)^{n-1}$

30. $\sum_{n=1}^{\infty} 3(0.5)^{n-1}$

31. $\sum_{n=1}^{\infty} (1.5)(0.25)^{n-1}$

32. **CHILD'S PLAY** Kimimela's little sister likes to swing at the playground. Yesterday, Kimimela pulled the swing back and let it go. The swing traveled a distance of 9 feet before heading back the other way. Each swing afterward was only 70% as long as the previous one. Find the total distance the swing traveled.



GEOMETRY **For Exercises 33 and 34, refer to square *ABCD*, which has a perimeter of 40 centimeters.**
If the midpoints of the sides are connected, a smaller square results. Suppose the process of connecting midpoints of sides and drawing new squares is continued indefinitely.

33. Write an infinite geometric series to represent the sum of the perimeters of all of the squares.

34. Find the sum of the perimeters of all of the squares.

35. **AVIATION** A hot-air balloon rises 90 feet in its first minute of flight. In each succeeding minute, it rises only 90% as far as it did during the preceding minute. What is the final height of the balloon?

More About. . .

Aviation

The largest hot-air balloon ever flown had a capacity of 2.6 million cubic feet.

Source: *The Guinness Book of Records*

36. The sum of an infinite geometric series is 81, and its common ratio is $\frac{2}{3}$. Find the first three terms of the series.

37. The sum of an infinite geometric series is 125, and the value of r is 0.4. Find the first three terms of the series.

38. The common ratio of an infinite geometric series is $\frac{11}{16}$, and its sum is $76\frac{4}{5}$. Find the first four terms of the series.

39. The first term of an infinite geometric series is -8, and its sum is $-13\frac{1}{3}$. Find the first four terms of the series.

Write each repeating decimal as a fraction.

40. $0.\overline{7}$ **41.** $0.\overline{1}$ **42.** $0.\overline{36}$ **43.** $0.\overline{82}$

44. $0.\overline{246}$ **45.** $0.\overline{427}$ **46.** $0.\overline{45}$ **47.** $0.2\overline{31}$

48. **CRITICAL THINKING** Derive the formula for the sum of an infinite geometric series by using the technique in Lessons 11-2 and 11-4. That is, write an equation for the sum S of a general infinite geometric series, multiply each side of the equation by r, and subtract equations.

49. **WRITING IN MATH** Answer the question that was posed at the beginning of the lesson.

How does an infinite geometric series apply to a bouncing ball?

Include the following in your answer:

- some formulas you might expect to see on the chalkboard if the character in the comic strip really was discussing a bouncing ball, and
- an explanation of how to find the total distance traveled, both up and down, by the bouncing ball described at the beginning of Lesson 11-3.

Standardized Test Practice

50. What is the sum of an infinite geometric series with a first term of 6 and a common ratio of $\frac{1}{2}$?

Ⓐ 3 Ⓑ 4 Ⓒ 9 Ⓓ 12

51. $2 + \frac{2}{3} + \frac{2}{9} + \frac{2}{27} + \ldots =$

Ⓐ $\frac{3}{2}$ Ⓑ $\frac{80}{27}$ Ⓒ 3 Ⓓ does not exist

www.algebra2.com/self_check_quiz

Maintain Your Skills

Mixed Review

Find S_n for each geometric series described. *(Lesson 11-4)*

52. $a_1 = 1, a_6 = -243, r = -3$

53. $a_1 = 72, r = \frac{1}{3}, n = 7$

54. PHYSICS A vacuum pump removes 20% of the air from a container with each stroke of its piston. What percent of the original air remains after five strokes of the piston? *(Lesson 11-3)*

Solve each equation or inequality. Check your solution. *(Lesson 10-1)*

55. $6^x = 216$

56. $2^{2x} = \frac{1}{8}$

57. $3^{x-2} \geq 27$

Simplify each expression. *(Lesson 9-2)*

58. $\frac{-2}{ab} + \frac{5}{a^2}$

59. $\frac{1}{x-3} - \frac{2}{x+1}$

60. $\frac{1}{x^2 + 6x + 8} + \frac{3}{x+4}$

Write an equation for the circle that satisfies each set of conditions. *(Lesson 8-3)*

61. center (2, 4), radius 6

62. endpoints of a diameter at (7, 3) and (−1, −5)

Find all the zeros of each function. *(Lesson 7-5)*

63. $f(x) = 8x^3 - 36x^2 + 22x + 21$

64. $g(x) = 12x^4 + 4x^3 - 3x^2 - x$

Write a quadratic equation with the given roots. Write the equation in the form $ax^2 + bx + c = 0$, where a, b, and c are integers. *(Lesson 6-3)*

65. 6, −6

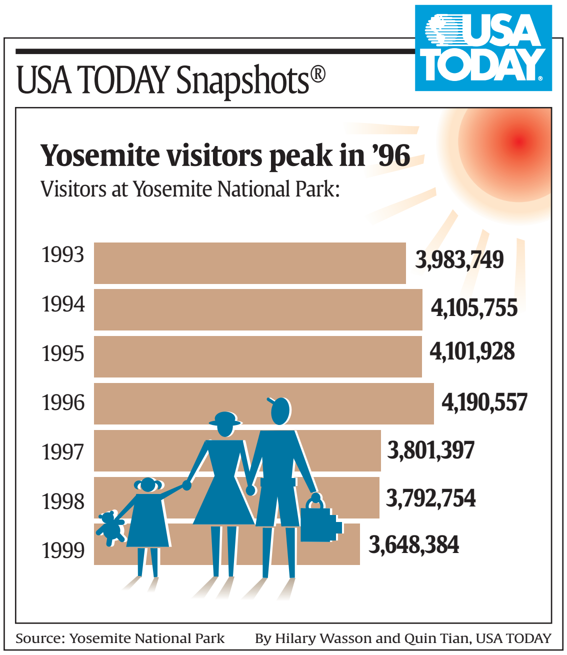
66. −2, −7

67. 6, 4

RECREATION **For Exercises 68 and 69, refer to the graph at the right.** *(Lesson 2-3)*

68. Find the average rate of change of the number of visitors to Yosemite National Park from 1996 to 1999.

69. Was the number of visitors increasing or decreasing from 1996 to 1999?



Getting Ready for the Next Lesson

PREREQUISITE SKILL **Find each function value.**
(To review ***evaluating functions****, see Lesson 2-1.)*

70. $f(x) = 2x, f(1)$

71. $g(x) = 3x - 3, g(2)$

72. $h(x) = -2x + 2, h(0)$

73. $f(x) = 3x - 1, f\left(\frac{1}{2}\right)$

74. $g(x) = x^2, g(2)$

75. $h(x) = 2x^2 - 4, h(0)$

Spreadsheet Investigation

Amortizing Loans

When a payment is made on a loan, part of the payment is used to cover the interest that has accumulated since the last payment. The rest is used to reduce the *principal*, or original amount of the loan. This process is called *amortization*. You can use a spreadsheet to analyze the payments, interest, and balance on a loan. A table that shows this kind of information is called an *amortization schedule*.

Example

Marisela just bought a new sofa for $495. The store is letting her make monthly payments of $43.29 at an interest rate of 9% for one year. How much will she still owe after six months?

Every month, the interest on the remaining balance will be $\frac{9\%}{12}$ or 0.75%. You can find the balance after a payment by multiplying the balance after the previous payment by 1 + 0.0075 or 1.0075 and then subtracting 43.29.

In a spreadsheet, use the column of numbers for the number of payments and use column B for the balance. Enter the interest rate and monthly payment in cells in column A so that they can be easily updated if the information changes.

The spreadsheet at the right shows the formulas for the balances after each of the first six payments. After six months, Marisela still owes $253.04.

Loans

	A	B
1	Interest rate	=495*(1+A2)-A5
2	0.0075	=B1*(1+A2)-A5
3		=B2*(1+A2)-A5
4	Monthly payment	=B3*(1+A2)-A5
5	43.29	=B4*(1+A2)-A5
6		=B5*(1+A2)-A5
7		

Sheet1 | Sh

Exercises

1. Let b_n be the balance left on Marisela's loan after n months. Write an equation relating b_n and b_{n+1}.

2. Payments at the beginning of a loan go more toward interest than payments at the end. What percent of Marisela's loan remains to be paid after half a year?

3. Extend the spreadsheet to the whole year. What is the balance after 12 payments? Why is it not 0?

4. Suppose Marisela decides to pay $50 every month. How long would it take her to pay off the loan?

5. Suppose that, based on how much she can afford, Marisela will pay a variable amount each month in addition to the $43.29. Explain how the flexibility of a spreadsheet can be used to adapt to this situation.

6. Jamie has a three-year, $12,000 car loan. The annual interest rate is 6%, and his monthly payment is $365.06. After twelve months, he receives an inheritance which he wants to use to pay off the loan. How much does he owe at that point?

11-6 Recursion and Special Sequences

What You'll Learn

- Recognize and use special sequences.
- Iterate functions.

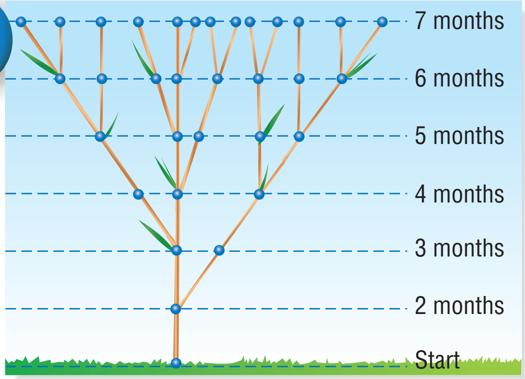
Vocabulary

- Fibonacci sequence
- recursive formula
- iteration

How is the Fibonacci sequence illustrated in nature?

A shoot on a sneezewort plant must grow for two months before it is strong enough to put out another shoot. After that, it puts out at least one shoot every month.

Month	1	2	3	4	5
Shoots	1	1	2	3	5



SPECIAL SEQUENCES Notice that the sequence 1, 1, 2, 3, 5, 8, 13, … has a pattern. Each term in the sequence is the sum of the two previous terms. For example, $8 = 3 + 5$ and $13 = 5 + 8$. This sequence is called the **Fibonacci sequence**, and it is found in many places in nature.

first term	a_1		1
second term	a_2		1
third term	a_3	$a_1 + a_2$	$1 + 1 = 2$
fourth term	a_4	$a_2 + a_3$	$1 + 2 = 3$
fifth term	a_5	$a_3 + a_4$	$2 + 3 = 5$
⋮	⋮	⋮	
nth term	a_n	$a_{n-2} + a_{n-1}$	

Study Tip

Reading Math

A recursive formula is often called a *recursive relation* or a *recurrence relation*.

The formula $a_n = a_{n-2} + a_{n-1}$ is an example of a **recursive formula**. This means that each term is formulated from one or more previous terms. To be able to use a recursive formula, you must be given the value(s) of the first term(s) so that you can start the sequence and then use the formula to generate the rest of the terms.

Example 1 Use a Recursive Formula

Find the first five terms of the sequence in which $a_1 = 4$ and $a_{n+1} = 3a_n - 2$, $n \geq 1$.

$a_{n+1} = 3a_n - 2$ Recursive formula

$a_{1+1} = 3a_1 - 2$ $n = 1$

$a_2 = 3(4) - 2$ or 10 $a_1 = 4$

$a_{2+1} = 3a_2 - 2$ $n = 2$

$a_3 = 3(10) - 2$ or 28 $a_2 = 10$

$a_{3+1} = 3a_3 - 2$ $n = 3$

$a_4 = 3(28) - 2$ or 82 $a_3 = 28$

$a_4 + 1 = 3a_4 - 2$ $n = 4$

$a_5 = 3(82) - 2$ or 244 $a_4 = 82$

The first five terms of the sequence are 4, 10, 28, 82, and 244.

Example 2 Find and Use a Recursive Formula

GARDENING **Mr. Yazaki discovered that there were 225 dandelions in his garden on the first Saturday of spring. He had time to pull out 100, but by the next Saturday, there were twice as many as he had left. Each Saturday in spring, he removed 100 dandelions, only to find that the number of remaining dandelions had doubled by the following Saturday.**

a. Write a recursive formula for the number of dandelions Mr. Yazaki finds in his garden each Saturday.

Let d_n represent the number of dandelions at the beginning of the nth Saturday. Mr. Yazaki will pull 100 of these out of his garden, leaving $d_n - 100$. The number d_{n+1} of dandelions the next Saturday will be twice this number. So, $d_{n+1} = 2(d_n - 100)$ or $2d_n - 200$.

b. Find the number of dandelions Mr. Yazaki would find on the fifth Saturday.

On the first Saturday, there were 225 dandelions, so $d_1 = 225$.

$d_{n+1} = 2d_n - 200$ Recursive formula

$d_{1+1} = 2d_1 - 200$ $n = 1$
$d_2 = 2(225) - 200$ or 250

$d_{2+1} = 2d_2 - 200$ $n = 2$
$d_3 = 2(250) - 200$ or 300

$d_{3+1} = 2d_3 - 200$ $n = 3$
$d_4 = 2(300) - 200$ or 400

$d_{4+1} = 2d_4 - 200$ $n = 4$
$d_5 = 2(400) - 200$ or 600

On the fifth Saturday, there would be 600 dandelions in Mr. Yazaki's garden.

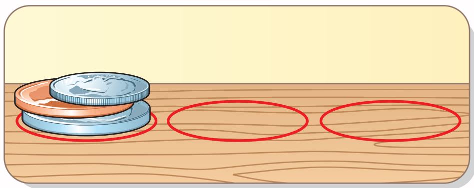
You can use sequences to analyze some games.

Algebra Activity

Special Sequences

The object of the *Towers of Hanoi* game is to move a stack of n coins from one position to another in the fewest number a_n of moves with these rules.

- You may only move one coin at a time.
- A coin must be placed on top of another coin, not underneath.
- A smaller coin may be placed on top of a larger coin, but not vice versa. For example, a penny may not be placed on top of a dime.



Model and Analyze

1. Draw three circles on a sheet of paper, as shown above. Place a penny on the first circle. What is the least number of moves required to get the penny to the second circle?
2. Place a nickel and a penny on the first circle, with the penny on top. What is the least number of moves that you can make to get the stack to another circle? (Remember, a nickel cannot be placed on top of a penny.)
3. Place a nickel, penny, and dime on the first circle. What is the least number of moves that you can take to get the stack to another circle?

Make a Conjecture

4. Place a quarter, nickel, penny, and dime on the first circle. Experiment to find the least number of moves needed to get the stack to another circle. Make a conjecture about a formula for the minimum number a_n of moves required to move a stack of n coins.

Study Tip

Look Back
To review **composition of functions**, see Lesson 7-7.

ITERATION **Iteration** is the process of composing a function with itself repeatedly. For example, if you compose a function with itself once, the result is $f \circ f(x)$ or $f(f(x))$. If you compose a function with itself two times, the result is $f \circ f \circ f(x)$ or $f(f(f(x)))$, and so on.

You can use iteration to recursively generate a sequence. Start with an initial value x_0. Let $x_1 = f(x_0)$, $x_2 = f(x_1)$ or $f(f(x_0))$, $x_3 = f(x_2)$ or $f(f(f(x_0)))$, and so on.

Example 3 Iterate a Function

Find the first three iterates x_1, x_2, x_3 of the function $f(x) = 2x + 3$ for an initial value of $x_0 = 1$.

To find the first iterate x_1, find the value of the function for $x_0 = 1$.

$x_1 = f(x_0)$ Iterate the function.
$= f(1)$ $x_0 = 1$
$= 2(1) + 3$ or 5 Simplify.

To find the second iterate x_2, substitute x_1 for x.

$x_2 = f(x_1)$ Iterate the function.
$= f(5)$ $x_1 = 5$
$= 2(5) + 3$ or 13 Simplify.

Substitute x_2 for x to find the third iterate.

$x_3 = f(x_2)$ Iterate the function.
$= f(13)$ $x_2 = 13$
$= 2(13) + 3$ or 29 Simplify.

Therefore, 1, 5, 13, 29 is an example of a sequence generated using iteration.

Check for Understanding

Concept Check

1. **Write** recursive formulas for the nth terms of arithmetic and geometric sequences.
2. **OPEN ENDED** Write a recursive formula for a sequence whose first three terms are 1, 1, and 3.
3. **State** whether the statement $x_n \neq x_{n-1}$ is *sometimes*, *always*, or *never* true if $x_n = f(x_{n-1})$. Explain.

Guided Practice

Find the first five terms of each sequence.

4. $a_1 = 12, a_{n+1} = a_n - 3$
5. $a_1 = -3, a_{n+1} = a_n + n$
6. $a_1 = 0, a_{n+1} = -2a_n - 4$
7. $a_1 = 1, a_2 = 2, a_{n+2} = 4a_{n+1} - 3a_n$

Find the first three iterates of each function for the given initial value.

8. $f(x) = 3x - 4, x_0 = 3$
9. $f(x) = -2x + 5, x_0 = 2$
10. $f(x) = x^2 + 2, x_0 = -1$

Application

BANKING **For Exercises 11 and 12, use the following information.**
Rita has deposited $1000 in a bank account. At the end of each year, the bank posts interest to her account in the amount of 5% of the balance, but then takes out a $10 annual fee.

11. Let b_0 be the amount Rita deposited. Write a recursive equation for the balance b_n in her account at the end of n years.
12. Find the balance in the account after four years.

Practice and Apply

Homework Help

For Exercises	See Examples
13–30	1–2
31–39	3

Extra Practice

See page 853.

Find the first five terms of each sequence.

13. $a_1 = -6, a_{n+1} = a_n + 3$

14. $a_1 = 13, a_{n+1} = a_n + 5$

15. $a_1 = 2, a_{n+1} = a_n - n$

16. $a_1 = 6, a_{n+1} = a_n + n + 3$

17. $a_1 = 9, a_{n+1} = 2\,a_n - 4$

18. $a_1 = 4, a_{n+1} = 3a_n - 6$

19. $a_1 = -1, a_2 = 5, a_{n+1} = a_n + a_{n-1}$

20. $a_1 = 4, a_2 = -3, a_{n+2} = a_{n+1} + 2a_n$

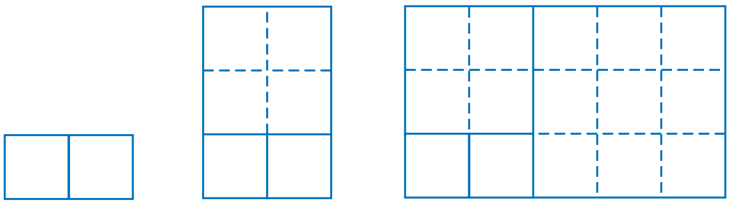
21. $a_1 = \frac{7}{2}, a_{n+1} = \frac{n}{n+1} \cdot a_n$

22. $a_1 = \frac{3}{4}, a_{n+1} = \frac{n^2+1}{n} \cdot a_n$

23. If $a_0 = 7$ and $a_{n+1} = a_n + 12$ for $n \geq 0$, find the value of a_5.

24. If $a_0 = 1$ and $a_{n+1} = -2.1$ for $n \geq 0$, then what is the value of a_4?

GEOMETRY **For Exercises 25 and 26, use the following information.**
Join two 1-unit by 1-unit squares to form a rectangle. Next, draw a larger square along a long side of the rectangle. Continue this process of drawing a square along a long side of the rectangle formed at the previous step.



Step 1 Step 2 Step 3

25. Write the sequence of the lengths of the sides of the squares you added at each step. Begin the sequence with the lengths of the sides of the two original squares.

26. Identify the sequence in Exercise 25.

27. **LOANS** The Cruz family is taking out a mortgage loan for $100,000 to buy a house. Their monthly payment is $678.79. The recursive formula $b_n = 1.006\,b_{n-1} - 678.79$ describes the balance left on the loan after n payments. Find the balances of the loan after each of the first eight payments.

GEOMETRY **For Exercises 28–30, study the triangular numbers shown below.**

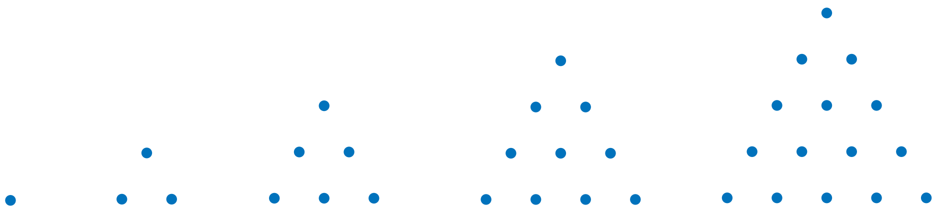


Figure 1 Figure 2 Figure 3 Figure 4 Figure 5

28. Write a sequence of the first five triangular numbers.

29. Write a recursive formula for the nth triangular number t_n.

30. What is the 200th triangular number?

Career Choices

Real Estate Agent

Most real estate agents are independent business-people who earn their income from commission.

Online Research

To learn more about a career in real estate, visit: www.algebra2.com/careers

Find the first three iterates of each function for the given initial value.

31. $f(x) = 9x - 2, x_0 = 2$

32. $f(x) = 4x - 3, x_0 = 2$

33. $f(x) = 3x + 5, x_0 = -4$

34. $f(x) = 5x + 1, x_0 = -1$

35. $f(x) = 2x^2 - 5, x_0 = -1$

36. $f(x) = 3x^2 - 4, x_0 = 1$

37. $f(x) = 2x^2 + 2x + 1, x_0 = \frac{1}{2}$

38. $f(x) = 3x^2 - 3x + 2, x_0 = \frac{1}{3}$

39. ECONOMICS If the rate of inflation is 2%, the cost of an item in future years can be found by iterating the function $c(x) = 1.02x$. Find the cost of a $70 portable stereo in four years if the rate of inflation remains constant.

40. CRITICAL THINKING Are there a function $f(x)$ and an initial value x_0 such that the first three iterates, in order, are 4, 4, and 7? If so, state such a function and initial value. If not, explain.

41. WRITING IN MATH Answer the question that was posed at the beginning of the lesson.

How is the Fibonacci sequence illustrated in nature?

Include the following in your answer:

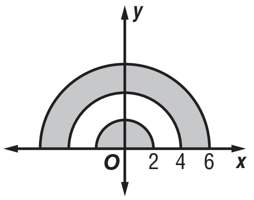
- the 13th term in the Fibonacci sequence, with an explanation of what it tells you about the plant described, and
- an explanation of why the Fibonacci sequence is neither arithmetic nor geometric.

Standardized Test Practice

42. If a is positive, what percent of $4a$ is 8?

(A) $\frac{a}{100}\%$ (B) $\frac{a}{2}\%$ (C) $\frac{8}{a}\%$ (D) $\frac{200}{a}\%$

43. The figure at the right is made of three concentric semicircles. What is the total area of the shaded regions?



(A) 4π units^2 (B) 10π units^2

(C) 12π units^2 (D) 20π units^2

Maintain Your Skills

Mixed Review

Find the sum of each infinite geometric series, if it exists. *(Lesson 11-5)*

44. $9 + 6 + 4 + \dots$

45. $\frac{1}{8} + \frac{1}{32} + \frac{1}{128} + \dots$

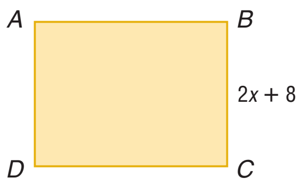
46. $4 - \frac{8}{3} + \frac{16}{9} + \dots$

Find the sum of each geometric series. *(Lesson 11-4)*

47. $2 - 10 + 50 - \dots$ to 6 terms

48. $3 + 1 + \frac{1}{3} + \dots$ to 7 terms

49. GEOMETRY The area of rectangle $ABCD$ is $6x^2 + 38x + 56$ square units. Its width is $2x + 8$ units. What is the length of the rectangle? *(Lesson 5-3)*



Getting Ready for the Next Lesson

BASIC SKILL **Evaluate each expression.**

50. $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$

51. $7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$

52. $\frac{4 \cdot 3}{2 \cdot 1}$

53. $\frac{6 \cdot 5 \cdot 4}{3 \cdot 2 \cdot 1}$

54. $\frac{9 \cdot 8 \cdot 7 \cdot 6}{4 \cdot 3 \cdot 2 \cdot 1}$

55. $\frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}$

Algebra Activity

A Follow-Up of Lesson 11-6

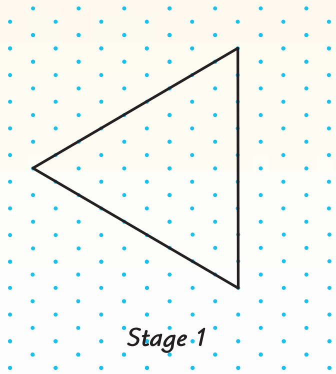
Fractals

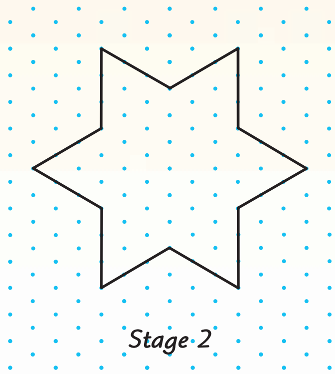
Fractals are sets of points that often involve intricate geometric shapes. Many fractals have the property that when small parts are magnified, the detail of the fractal is not lost. In other words, the magnified part is made up of smaller copies of itself. Such fractals can be constructed recursively.

You can use isometric dot paper to draw stages of the construction of a fractal called the *von Koch snowflake*.

Stage 1 Draw an equilateral triangle with sides of length 9 units on the dot paper.

Stage 2 Now remove the middle third of each side of the triangle from Stage 1 and draw the other two sides of an equilateral triangle pointing outward.





Imagine continuing this process indefinitely. The von Koch snowflake is the shape that these stages approach.

Model and Analyze

1. Copy and complete the table. Draw Stage 3, if necessary.

Stage	1	2	3	4
Number of Segments	3	12		
Length of each Segment	9	3		
Perimeter	27	36		

2. Write recursive formulas for the number s_n of segments in Stage n, the length ℓ_n of each segment in Stage n, and the perimeter P_n of Stage n.

3. Write nonrecursive formulas for s_n, ℓ_n, and P_n.

4. What is the perimeter of the von Koch snowflake? Explain.

5. Explain why the area of the von Koch snowflake can be represented by the infinite series $\frac{81\sqrt{3}}{4} + \frac{27\sqrt{3}}{4} + 3\sqrt{3} + \frac{4\sqrt{3}}{3} + \ldots$.

6. Find the sum of the series in Exercise 5. Explain your steps.

7. Do you think the results of Exercises 4 and 6 are contradictory? Explain.

CONTENTS

11-7 The Binomial Theorem

What You'll Learn

- Use Pascal's triangle to expand powers of binomials.
- Use the Binomial Theorem to expand powers of binomials.

How does a power of a binomial describe the numbers of boys and girls in a family?

According to the U.S. Census Bureau, ten percent of families have three or more children. If a family has four children, there are six sequences of births of boys and girls that result in two boys and two girls. These sequences are listed below.

BBGG   BGBG   BGGB   GBBG   GBGB   GGBB

Vocabulary

- Pascal's triangle
- Binomial Theorem
- factorial

PASCAL'S TRIANGLE You can use the coefficients in powers of binomials to count the number of possible sequences in situations such as the one above. Remember that a binomial is a polynomial with two terms. Expand a few powers of the binomial $b + g$.

$$(b+g)^0 = 1b^0g^0$$
$$(b+g)^1 = 1b^1g^0 + 1b^0g^1$$
$$(b+g)^2 = 1b^2g^0 + 2b^1g^1 + 1b^0g^1$$
$$(b+g)^3 = 1b^3g^0 + 3b^2g^1 + 3b^1g^2 + 1b^0g^3$$
$$(b+g)^4 = 1b^4g^0 + 4b^3g^1 + 6b^2g^2 + 4b^1g^3 + 1b^0g^4$$

The coefficient 6 of the b^2g^2 term in the expansion of $(b + g)^4$ gives the number of sequences of births that result in two boys and two girls. As another example, the coefficient 4 of the b^1g^3 term gives the number of sequences with one boy and 3 girls.

Here are some patterns that can be seen in any binomial expansion of the form $(a + b)^n$.

1. There are $n + 1$ terms.
2. The exponent n of $(a + b)^n$ is the exponent of a in the first term and the exponent of b in the last term.
3. In successive terms, the exponent of a decreases by one, and the exponent of b increases by one.
4. The sum of the exponents in each term is n.
5. The coefficients are symmetric. They increase at the beginning of the expansion and decrease at the end.

The coefficients form a pattern that is often displayed in a triangular formation. This is known as **Pascal's triangle**. Notice that each row begins and ends with 1. Each coefficient is the sum of the two coefficients above it in the previous row.

$$
\begin{array}{ll}
(a+b)^0 & \quad\quad\quad\quad\quad 1 \\
(a+b)^1 & \quad\quad\quad\quad 1 \quad 1 \\
(a+b)^2 & \quad\quad\quad 1 \quad 2 \quad 1 \\
(a+b)^3 & \quad\quad 1 \quad 3 \quad 3 \quad 1 \\
(a+b)^4 & \quad 1 \quad 4 \quad 6 \quad 4 \quad 1 \\
(a+b)^5 & 1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1
\end{array}
$$

More About . . .

Pascal's Triangle

Although he did not discover it, Pascal's triangle is named for the French mathematician Blaise Pascal (1623–1662).

Example 1 Use Pascal's Triangle

Expand $(x + y)^7$.

Write two more rows of Pascal's triangle.

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

Use the patterns of a binomial expansion and the coefficients to write the expansion of $(x + y)^7$.

$$(x + y)^7 = 1x^7y^0 + 7x^6y^1 + 21x^5y^2 + 35x^4y^3 + 35x^3y^4 + 21x^2y^5 + 7x^1y^6 + 1x^0y^7$$
$$= x^7 + 7x^6y + 21x^5y^2 + 35x^4y^3 + 35x^3y^4 + 21x^2y^5 + 7xy^6 + y^7$$

THE BINOMIAL THEOREM

Another way to show the coefficients in a binomial expansion is to write them in terms of the previous coefficients.

$(a + b)^0$	1				
$(a + b)^1$	1	$\frac{1}{1}$			
$(a + b)^2$	1	$\frac{2}{1}$	$\frac{2 \cdot 1}{1 \cdot 2}$		
$(a + b)^3$	1	$\frac{3}{1}$	$\frac{3 \cdot 2}{1 \cdot 2}$	$\frac{3 \cdot 2 \cdot 1}{1 \cdot 2 \cdot 3}$	
$(a + b)^4$	1	$\frac{4}{1}$	$\frac{4 \cdot 3}{1 \cdot 2}$	$\frac{4 \cdot 3 \cdot 2}{1 \cdot 2 \cdot 3}$	$\frac{4 \cdot 3 \cdot 2 \cdot 1}{1 \cdot 2 \cdot 3 \cdot 4}$

Eliminate common factors that are shown in color.

This pattern provides the coefficients of $(a + b)^n$ for any nonnegative integer n. The pattern is summarized in the **Binomial Theorem**.

Key Concept — Binomial Theorem

If n is a nonnegative integer, then

$$(a + b)^n = 1a^nb^0 + \frac{n}{1}a^{n-1}b^1 + \frac{n(n-1)}{1 \cdot 2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}a^{n-3}b^3 + \dots + 1a^0b^n.$$

Example 2 Use the Binomial Theorem

Expand $(a - b)^6$.

The expansion will have seven terms. Use the sequence $1, \frac{6}{1}, \frac{6 \cdot 5}{1 \cdot 2}, \frac{6 \cdot 5 \cdot 4}{1 \cdot 2 \cdot 3}$ to find the coefficients for the first four terms. Then use symmetry to find the remaining coefficients.

$$(a - b)^6 = 1a^6(-b)^0 + \frac{6}{1}a^5(-b)1 + \frac{6 \cdot 5}{1 \cdot 2}a^4(-b)^2 + \frac{6 \cdot 5 \cdot 4}{1 \cdot 2 \cdot 3}a^3(-b)^3 + \dots + 1a^0(-b)^6$$
$$= a^6 - 6a^5b + 15a^4b^2 - 20a^3b^3 + 15a^2b^4 - 6ab^5 + b^6$$

Notice that in terms having the same coefficients, the exponents are reversed, as in $15a^4b^2$ and $15a^2b^4$.

Study Tip

Graphing Calculators

On a TI-83 Plus, the factorial symbol, !, is located on the MATH PRB menu.

The factors in the coefficients of binomial expansions involve special products called **factorials**. For example, the product $4 \cdot 3 \cdot 2 \cdot 1$ is written $4!$ and is read *4 factorial*. In general, if n is a positive integer, then $n! = n(n - 1)(n - 2)(n - 3) \dots 2 \cdot 1$. *By definition, $0! = 1$.*

www.algebra2.com/extra_examples

CONTENTS

Example 3 Factorials

Evaluate $\frac{8!}{3!5!}$.

$$\frac{8!}{3!5!} = \frac{8 \cdot 7 \cdot 6 \cdot \cancel{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}^{1}}{3 \cdot 2 \cdot 1 \cdot \cancel{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}_{1}}$$

Note that $8! = 8 \cdot 7 \cdot 6 \cdot 5!$, so $\frac{8!}{3!5!} = \frac{8 \cdot 7 \cdot 6 \cdot 5!}{3!5!}$ or $\frac{8 \cdot 7 \cdot 6}{3 \cdot 2 \cdot 1}$

$$= \frac{8 \cdot 7 \cdot 6}{3 \cdot 2 \cdot 1} \text{ or } 56$$

An expression such as $\frac{6 \cdot 5 \cdot 4}{1 \cdot 2 \cdot 3}$ in Example 2 can be written as a quotient of factorials. In this case, $\frac{6 \cdot 5 \cdot 4}{1 \cdot 2 \cdot 3} = \frac{6!}{3!3!}$. Using this idea, you can rewrite the expansion of $(a + b)^6$ using factorials.

$$(a + b)^6 = \frac{6!}{6!0!}a^6b^0 + \frac{6!}{5!1!}a^5b^1 + \frac{6!}{4!2!}a^4b^2 + \frac{6!}{3!3!}a^3b^3 + \frac{6!}{2!4!}a^2b^4 + \frac{6!}{1!5!}a^1b^5 + \frac{6!}{0!6!}a^0b^6$$

You can also write this series using sigma notation.

$$(a + b)^6 = \sum_{k=0}^{6} \frac{6!}{(6 - k)!k!}a^{6 - k}b^k$$

In general, the Binomial Theorem can be written both in factorial notation and in sigma notation.

Study Tip

Missing Steps

If you don't understand a step like $\frac{6 \cdot 5 \cdot 4}{1 \cdot 2 \cdot 3} = \frac{6!}{3!3!}$, work it out on a piece of scrap paper.

$$\frac{6 \cdot 5 \cdot 4}{1 \cdot 2 \cdot 3} = \frac{6 \cdot 5 \cdot 4 \cdot 3!}{1 \cdot 2 \cdot 3 \cdot 3!} = \frac{6!}{3!3!}$$

Key Concept — Binomial Theorem, Factorial Form

$$(a + b)^n = \frac{n!}{n!0!}a^nb^0 + \frac{n!}{(n - 1)!1!}a^{n - 1}b^1 + \frac{n!}{(n - 2)!2!}a^{n - 2}b^2 + \ldots + \frac{n!}{0!n!}a^0b^n$$

$$= \sum_{k=0}^{n} \frac{n!}{(n - k)!k!}a^{n - k}b^k$$

Example 4 Use a Factorial Form of the Binomial Theorem

Expand $(2x + y)^5$.

$$(2x + y)^5 = \sum_{k=0}^{5} \frac{5!}{(5 - k)!k!}(2x)^{5-k}y^k$$ Binomial Theorem, factorial form

$$= \frac{5!}{5!0!}(2x)^5y^0 + \frac{5!}{4!1!}(2x)^4y^1 + \frac{5!}{3!2!}(2x)^3y^2 + \frac{5!}{2!3!}(2x)^2y^3 + \frac{5!}{1!4!}(2x)^1y^4 + \frac{5!}{0!5!}(2x)^0y^5$$ Let $k = 0, 1, 2, 3, 4,$ and 5.

$$= \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot 1}(2x)^5 + \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{4 \cdot 3 \cdot 2 \cdot 1 \cdot 1}(2x)^4 y + \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1 \cdot 2 \cdot 1}(2x)^3 y^2 + \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 1 \cdot 3 \cdot 2 \cdot 1}(2x)^2 y^3 + \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{1 \cdot 4 \cdot 3 \cdot 2 \cdot 1}(2x)y^4 + \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{1 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}y^5$$

$$= 32x^5 + 80x^4y + 80x^3y^2 + 40x^2y^3 + 10xy^4 + y^5$$ Simplify.

Sometimes you need to know only a particular term of a binomial expansion. Note that when the Binomial Theorem is written in sigma notation, $k = 0$ for the first term, $k = 1$ for the second term, and so on. In general, the value of k is always one less than the number of the term you are finding.

Example 5 Find a Particular Term

Find the fifth term in the expansion of $(p + q)^{10}$.

First, use the Binomial Theorem to write the expansion in sigma notation.

$$(p+q)^{10} = \sum_{k=0}^{10} \frac{10!}{(10-k)!k!} p^{10-k} q^k$$

In the fifth term, $k = 4$.

$$\frac{10!}{(10-k)!k!} p^{10-k} q^k = \frac{10!}{(10-4)!4!} p^{10-4} q^4 \quad k = 4$$

$$= \frac{10 \cdot 9 \cdot 8 \cdot 7}{4 \cdot 3 \cdot 2 \cdot 1} p^6 q^4 \quad \frac{10!}{6!4!} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6!}{6!4!} \text{ or } \frac{10 \cdot 9 \cdot 8 \cdot 7}{4 \cdot 3 \cdot 2 \cdot 1}$$

$$= 210p^6q^4 \quad \text{Simplify.}$$

Check for Understanding

Concept Check

1. **List** the coefficients in the row of Pascal's triangle corresponding to $n = 8$.
2. **Identify** the coefficient of $a^{n-1}b$ in the expansion of $(a + b)^n$.
3. **OPEN ENDED** Write a power of a binomial for which the first term of the expansion is $625x^4$.

Guided Practice

Evaluate each expression.

4. $8!$
5. $\frac{13!}{9!}$
6. $\frac{12!}{2!10!}$

Expand each power.

7. $(p + q)^5$
8. $(t + 2)^6$
9. $(x - 3y)^4$

Find the indicated term of each expansion.

10. fourth term of $(a + b)^8$
11. fifth term of $(2a + 3b)^{10}$

Application

12. **SCHOOL** Mr. Hopkins is giving a five-question true-false quiz. How many ways could a student answer the questions with three trues and two falses?

Practice and Apply

Homework Help

For Exercises	See Examples
13–18	3
19–33	1, 2, 4
34–41	5

Extra Practice

See page 853.

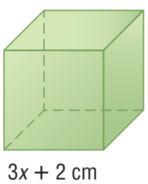
Evaluate each expression.

13. $9!$
14. $13!$
15. $\frac{9!}{7!}$
16. $\frac{7!}{4!}$
17. $\frac{12!}{8!4!}$
18. $\frac{14!}{5!9!}$

Expand each power.

19. $(a - b)^3$
20. $(m + n)^4$
21. $(r + s)^8$
22. $(m - a)^5$
23. $(x + 3)^5$
24. $(a - 2)^4$
25. $(2b - x)^4$
26. $(2a + b)^6$
27. $(3x - 2y)^5$
28. $(3x + 2y)^4$
29. $\left(\frac{a}{2} + 2\right)^5$
30. $\left(3 + \frac{m}{3}\right)^5$

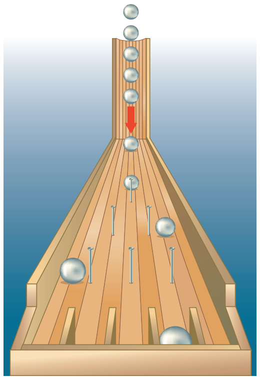
31. **GEOMETRY** Write an expanded expression for the volume of the cube at the right.



WebQuest

Pascal's triangle displays many patterns. Visit www.algebra2.com/webquest to continue work on your WebQuest project.

32. **GAMES** The diagram shows the board for a game in which ball bearings are dropped down a chute. A pattern of nails and dividers causes the bearings to take various paths to the sections at the bottom. For each section, how many paths through the board lead to that section?



33. **INTRAMURALS** Ofelia is taking ten shots in the intramural free-throw shooting competition. How many sequences of makes and misses are there that result in her making eight shots and missing two?

Find the indicated term of each expansion.

34. sixth term of $(x - y)^9$

35. seventh term of $(x + y)^{12}$

36. fourth term of $(x + 2)^7$

37. fifth term of $(a - 3)^8$

38. fifth term of $(2a + 3b)^{10}$

39. fourth term of $(2x + 3y)^9$

40. fourth term of $\left(x + \frac{1}{3}\right)^7$

41. sixth term of $\left(x - \frac{1}{2}\right)^{10}$

42. **CRITICAL THINKING** Explain why $\frac{12!}{7!5!} + \frac{12!}{6!6!} = \frac{13!}{7!6!}$ without finding the value of any of the expressions.

43. **WRITING IN MATH** Answer the question that was posed at the beginning of the lesson.

How does a power of a binomial describe the numbers of boys and girls in a family?

Include the following in your answer:

- the expansion of $(b + g)^5$ and what it tells you about sequences of births of boys and girls in families with five children, and
- an explanation of how to find a formula for the number of sequences of births that have exactly k girls in a family of n children.

Standardized Test Practice

44. Which of the following represents the values of x that are solutions of the inequality $x^2 < x + 20$?

(A) $x > -4$

(B) $x < 5$

(C) $-5 < x < 4$

(D) $-4 < x < 5$

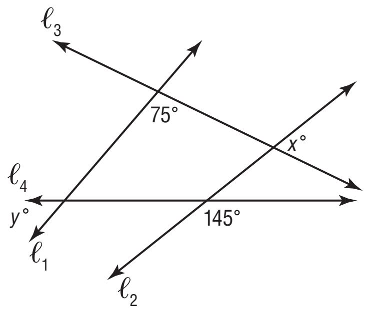
45. If four lines intersect as shown in the figure at the right, $x + y =$

(A) 70.

(B) 115.

(C) 140.

(D) It cannot be determined from the information given.



Maintain Your Skills

Mixed Review

Find the first five terms of each sequence. *(Lesson 11-6)*

46. $a_1 = 7, a_{n+1} = a_n - 2$

47. $a_1 = 3, a_{n+1} = 2a_n - 1$

48. CLOCKS The spring in Juanita's old grandfather clock is broken. When you try to set the pendulum in motion by holding it against the wall of the clock and letting go, it follows a swing pattern of 25 centimeters, 20 centimeters, 16 centimeters, and so on until it comes to rest. What is the total distance the pendulum swings before coming to rest? *(Lesson 11-5)*

Express each logarithm in terms of common logarithms. Then approximate its value to four decimal places. *(Lesson 10-4)*

49. $\log_2 5$

50. $\log_3 10$

51. $\log_5 8$

Determine any vertical asymptotes and holes in the graph of each rational function. *(Lesson 9-3)*

52. $f(x) = \frac{1}{x^2 + 5x + 6}$

53. $f(x) = \frac{x + 2}{x^2 + 3x - 4}$

54. $f(x) = \frac{x^2 + 4x + 3}{x + 3}$

Without writing the equation in standard form, state whether the graph of each equation is a *parabola*, *circle*, *ellipse*, or *hyperbola*. *(Lesson 8-6)*

55. $x^2 - 6x - y^2 - 3 = 0$

56. $4y - x + y^2 = 1$

Determine whether each pair of functions are inverse functions. *(Lesson 7-8)*

57. $f(x) = x + 3$
$g(x) = x - 3$

58. $f(x) = 2x + 1$
$g(x) = \frac{x + 1}{2}$

Getting Ready for the Next Lesson

PREREQUISITE SKILL **State whether each statement is *true* or *false* when $n = 1$. Explain.** *(To review **evaluating expressions**, see Lesson 1-1.)*

59. $1 = \frac{n(n + 1)}{2}$

60. $1 = \frac{(n + 1)(2n + 1)}{2}$

61. $1 = \frac{n^2(n + 1)^2}{4}$

62. $3^n - 1$ is even.

Practice Quiz 2

Lessons 11-4 through 11-7

Find the sum of each geometric series. *(Lessons 11-4 and 11-5)*

1. $a_1 = 5, r = 3, n = 12$

2. $\sum_{n-1}^{6} 2(-3)^{n-1}$

3. $\sum_{n=1}^{\infty} 8\left(\frac{2}{3}\right)^{n-1}$

4. $5 + 1 + \frac{1}{5} + \ldots$

Find the first five terms of each sequence. *(Lesson 11-6)*

5. $a_1 = 1, a_{n+1} = 2a_n + 3$

6. $a_1 = 2, a_{n+1} = a_n + 2n$

7. Find the first three iterates of the function $f(x) = -3x + 2$ for an initial value of $x_0 = -1$. *(Lesson 11-6)*

Expand each power. *(Lesson 11-7)*

8. $(3x + y)^5$

9. $(a + 2)^6$

10. Find the fifth term of the expansion of $(2a + b)^9$. *(Lesson 11-7)*

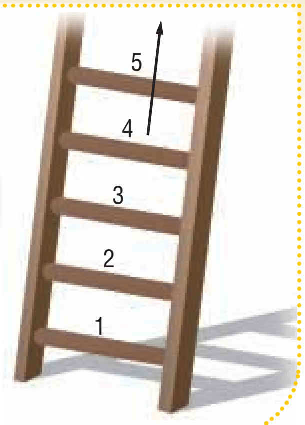
11-8 Proof and Mathematical Induction

What You'll Learn

- Prove statements by using mathematical induction.
- Disprove statements by finding a counterexample.

How does the concept of a ladder help you prove statements about numbers?

Imagine the positive integers as a ladder that goes upward forever. You know that you cannot leap to the top of the ladder, but you can stand on the first step, and no matter which step you are on, you can always climb one step higher. Is there any step you cannot reach?



Vocabulary

- mathematical induction
- inductive hypothesis

MATHEMATICAL INDUCTION

Mathematical induction is used to prove statements about positive integers. An induction proof consists of three steps.

Study Tip

Step 1

In many cases, it will be helpful to let $n = 1$.

Key Concept — Mathematical Induction

Step 1 Show that the statement is true for some integer n.

Step 2 Assume that the statement is true for some positive integer k, where $k \geq n$. This assumption is called the **inductive hypothesis**.

Step 3 Show that the statement is true for the next integer $k + 1$.

Example 1 Summation Formula

Prove that the sum of the squares of the first n positive integers is $\frac{n(n+1)(2n+1)}{6}$. That is, prove that $1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$.

Step 1 When $n = 1$, the left side of the given equation is 1^2 or 1. The right side is $\frac{1(1+1)[2(1)+1]}{6}$ or 1. Thus, the equation is true for $n = 1$.

Step 2 Assume $1^2 + 2^2 + 3^2 + \ldots + k^2 = \frac{k(k+1)(2k+1)}{6}$ for a positive integer k.

Step 3 Show that the given equation is true for $n = k + 1$.

$$1^2 + 2^2 + 3^2 + \ldots + k^2 + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \quad \text{Add } (k+1)^2 \text{ to each side.}$$

$$= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} \quad \text{Add.}$$

$$= \frac{(k+1)[k(2k+1) + 6(k+1)]}{6} \quad \text{Factor.}$$

$$= \frac{(k+1)[2k^2 + 7k + 6]}{6} \quad \text{Simplify.}$$

$$= \frac{(k+1)(k+2)(2k+3)}{6} \quad \text{Factor.}$$

$$= \frac{(k+1)[(k+1)+1][2(k+1)+1]}{6}$$

The last expression on page 618 is the right side of the equation to be proved, where n has been replaced by $k + 1$. Thus, the equation is true for $n = k + 1$.

This proves that $1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$ for all positive integers n.

Example 2 Divisibility

Prove that $7^n - 1$ is divisible by 6 for all positive integers n.

Step 1 When $n = 1$, $7^n - 1 = 7^1 - 1$ or 6. Since 6 is divisible by 6, the statement is true for $n = 1$.

Step 2 Assume that $7^k - 1$ is divisible by 6 for some positive integer k. This means that there is a whole number r such that $7^k - 1 = 6r$.

Step 3 Show that the statement is true for $n = k + 1$.

$7^k - 1 = 6r$ Inductive hypothesis

$7^k = 6r + 1$ Add 1 to each side.

$7(7^k) = 7(6r + 1)$ Multiply each side by 7.

$7^{k+1} = 42r + 7$ Simplify.

$7^{k+1} - 1 = 42r + 6$ Subtract 1 from each side.

$7^{k+1} - 1 = 6(7r + 1)$ Factor.

Since r is a whole number, $7r + 1$ is a whole number. Therefore, $7^{k+1} - 1$ is divisible by 6. Thus, the statement is true for $n = k + 1$.

This proves that $7^n - 1$ is divisible by 6 for all positive integers n.

Study Tip

Reading Math
One of the meanings of *counter* is *to oppose*, so a counterexample is an example that opposes a hypothesis.

COUNTEREXAMPLES Of course, not every formula that you can write is true. A formula that works for a few positive integers may not work for *every* positive integer. You can show that a formula is not true by finding a *counterexample*. This often involves trial and error.

Example 3 Counterexample

Find a counterexample for the formula $1^4 + 2^4 + 3^4 + \dots + n^4 = 1 + (4n - 4)^2$.

Check the first few positive integers.

n	Left Side of Formula	Right Side of Formula	
1	1^4 or 1	$1 + [4(1) - 4]^2 = 1 + 0^2$ or 1	true
2	$1^4 + 2^4 = 1 + 16$ or 17	$1 + [4(2) - 4]^2 = 1 + 4^2$ or 17	true
3	$1^4 + 2^4 + 3^4 = 1 + 16 + 81$ or 98	$1 + [4(3) - 4]^2 = 1 + 64$ or 65	false

The value $n = 3$ is a counterexample for the formula.

Check for Understanding

Concept Check

1. **Describe** some of the types of statements that can be proved by using mathematical induction.

2. **Explain** the difference between mathematical induction and a counterexample.

3. **OPEN ENDED** Write an expression of the form $b^n - 1$ that is divisible by 2 for all positive integers n.

www.algebra2.com/extra_examples

Guided Practice

Prove that each statement is true for all positive integers.

4. $1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$

5. $\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots + \frac{1}{2^n} = 1 - \frac{1}{2^n}$

6. $4^n - 1$ is divisible by 3.

7. $5^n + 3$ is divisible by 4.

Find a counterexample for each statement.

8. $1 + 2 + 3 + \ldots + n = n^2$

9. $2^n + 2n$ is divisible by 4.

Application

10. PARTIES Suppose that each time a new guest arrives at a party, he or she shakes hands with each person already at the party. Prove that after n guests have arrived, a total of $\frac{n(n-1)}{2}$ handshakes have taken place.

Practice and Apply

Homework Help

For Exercises	See Examples
11–23, 31	1
24	1, 2
25–30	3

Extra Practice

See page 853.

Prove that each statement is true for all positive integers.

11. $1 + 5 + 9 + \ldots + (4n - 3) = n(2n - 1)$

12. $2 + 5 + 8 + \ldots + (3n - 1) = \frac{n(3n+1)}{2}$

13. $1^3 + 2^3 + 3^3 + \ldots + n^3 = \frac{n^2(n+1)^2}{4}$

14. $1^2 + 3^2 + 5^2 + \ldots + (2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}$

15. $\frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \ldots + \frac{1}{3^n} = \frac{1}{2}\left(1 - \frac{1}{3^n}\right)$

16. $\frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \ldots + \frac{1}{4^n} = \frac{1}{3}\left(1 - \frac{1}{4^n}\right)$

17. $8^n - 1$ is divisible by 7.

18. $9^n - 1$ is divisible by 8.

19. $12^n + 10$ is divisible by 11.

20. $13^n + 11$ is divisible by 12.

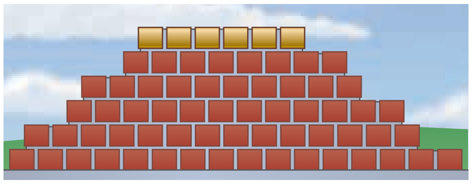
More About. . .

Architecture

The Vietnam Veterans Memorial lists the names of 58,220 deceased or missing soldiers.

Source: National Parks Service

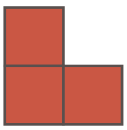
21. ARCHITECTURE A memorial being constructed in a city park will be a brick wall, with a top row of six gold-plated bricks engraved with the names of six local war veterans. Each row has two more bricks than the row above it. Prove that the number of bricks in the top n rows is $n^2 + 5n$.



22. GEOMETRIC SERIES Use mathematical induction to prove the formula $a_1 + a_1r + a_1r^2 + \ldots + a_1r^{n-1} = \frac{a_1(1 - r^n)}{1 - r}$ for the sum of a finite geometric series.

23. ARITHMETIC SERIES Use mathematical induction to prove the formula $a_1 + (a_1 + d) + (a_1 + 2d) + \ldots + [a_1 + (n-1)d] = \frac{n}{2}[2a_1 + (n-1)d]$ for the sum of an arithmetic series.

24. PUZZLES Show that a 2^n by 2^n checkerboard with the top right square missing can always be covered by nonoverlapping L-shaped tiles like the one at the right.



Find a counterexample for each statement.

25. $1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(3n-1)}{2}$

26. $1^3 + 3^3 + 5^3 + \ldots + (2n-1)^3 = 12n^3 - 23n^2 + 12n$

27. $3^n + 1$ is divisible by 4.

28. $2^n + 2n^2$ is divisible by 4.

29. $n^2 - n + 11$ is prime.

30. $n^2 + n + 41$ is prime.

31. **CRITICAL THINKING** Refer to Example 2. Explain how to use the Binomial Theorem to show that $7^n - 1$ is divisible by 6 for all positive integers n.

32. *WRITING IN MATH* Answer the question that was posed at the beginning of the lesson.

How does the concept of a ladder help you prove statements about numbers?

Include the following in your answer:

- an explanation of which part of an inductive proof corresponds to stepping onto the bottom step of the ladder, and
- an explanation of which part of an inductive proof corresponds to climbing from one step on the ladder to the next.

Standardized Test Practice

33. $\dfrac{x - \frac{4}{x}}{1 - \frac{4}{x} + \frac{4}{x^2}} =$

(A) $\frac{x}{x-2}$ (B) $\frac{x^2+2}{x-2}$ (C) $\frac{x^2+2x}{x-2}$ (D) $\frac{x^2+2x}{(x-2)^2}$

34. Quantitative Comparison

Compare the quantity in Column A and the quantity in Column B. Then determine whether:

(A) **the quantity in Column A is greater,**

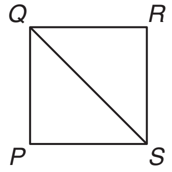
(B) **the quantity in Column B is greater,**

(C) **the two quantities are equal, or**

(D) **the relationship cannot be determined from the information given.**

PQRS is a square.

Column A	Column B
2	$\frac{\text{length of } \overline{QS}}{\text{length of } \overline{RS}}$



Maintain Your Skills

Mixed Review

Expand each power. *(Lesson 11-7)*

35. $(x + y)^6$ **36.** $(a - b)^7$ **37.** $(2x + y)^8$

Find the first three iterates of each function for the given initial value. *(Lesson 11-6)*

38. $f(x) = 3x - 2, x_0 = 2$ **39.** $f(x) = 4x^2 - 2, x_0 = 1$

40. **BIOLOGY** Suppose an amoeba divides into two amoebas once every hour. How long would it take for a single amoeba to become a colony of 4096 amoebas? *(Lesson 10-2)*

Solve each equation. Check your solutions. *(Lesson 9-6)*

41. $\frac{1}{y+1} - \frac{3}{y-3} = 2$ **42.** $\frac{6}{a-7} = \frac{a-49}{a^2-7a} + \frac{1}{a}$

Chapter 11

Study Guide and Review

Vocabulary and Concept Check

arithmetic means (p. 580)
arithmetic sequence (p. 578)
arithmetic series (p. 583)
Binomial Theorem (p. 613)
common difference (p. 578)
common ratio (p. 588)
factorial (p. 613)
Fibonacci sequence (p. 606)
geometric means (p. 590)
geometric sequence (p. 588)
geometric series (p. 594)
index of summation (p. 585)
inductive hypothesis (p. 618)
infinite geometric series (p. 599)
iteration (p. 608)
mathematical induction (p. 618)
partial sum (p. 599)
Pascal's triangle (p. 612)
recursive formula (p. 606)
sequence (p. 578)
series (p. 583)
sigma notation (p. 585)
term (p. 578)

Choose the term from the list above that best completes each statement.

1. A(n) ______________ of an infinite series is the sum of a certain number of terms.
2. If a sequence has a common ratio, then it is a(n) _______________.
3. Using __________, the series 2 + 5 + 8 + 11 + 14 can be written as $\sum_{n=1}^{5}(3n-1)$.
4. Eleven and 17 are the two ____ between 5 and 23 in the sequence 5, 11, 17, 23.
5. Using the __________, $(a-2)^4$ can be expanded to $a^4 - 8a^3 + 24a^2 - 32a + 16$.
6. The __________ of the sequence 3, 2, $\frac{4}{3}$, $\frac{8}{9}$, $\frac{16}{27}$ is $\frac{2}{3}$.
7. The ________ 11 + 16.5 + 22 + 27.5 + 33 has a sum of 110.
8. A(n) ____ is expressed as $n! = n(n-1)(n-2) \ldots 2 \cdot 1$.

Lesson-by-Lesson Review

11-1 Arithmetic Sequences

See pages 578–582.

Concept Summary

- An arithmetic sequence is formed by adding a constant to each term to get the next term.
- The nth term a_n of an arithmetic sequence with first term a_1 and common difference d is given by $a_n = a_1 + (n-1)d$, where n is any positive integer.

Examples

1 Find the 12th term of an arithmetic sequence if $a_1 = -17$ and $d = 4$.

$a_n = a_1 + (n-1)d$ — Formula for the nth term

$a_{12} = -17 + (12-1)4$ — $n = 12$, $a_1 = -17$, $d = 4$

$a_{12} = 27$ — Simplify.

2 Find the two arithmetic means between 4 and 25.

$a_n = a_1 + (n-1)d$ — Formula for the nth term

$a_4 = 4 + (4-1)d$ — $n = 4$, $a_1 = 4$

$25 = 4 + 3d$ — $a_4 = 25$

$7 = d$ — The arithmetic means are 4 + 7 or 11 and 11 + 7 or 18.

www.algebra2.com/vocabulary_review

Exercises Find the indicated term of each arithmetic sequence. *See Example 2 on p. 579.*

9. $a_1 = 6, d = 8, n = 5$
10. $a_1 = -5, d = 7, n = 22$
11. $a_1 = 5, d = -2, n = 9$
12. $a_1 = -2, d = -3, n = 15$

Find the arithmetic means in each sequence. *See Example 4 on page 580.*

13. -7, __?__, __?__, __?__, 9
14. 12, __?__, __?__, 4
15. 9, __?__, __?__, __?__, __?__, -6
16. 56, __?__, __?__, __?__, 28

11-2 Arithmetic Series

See pages 583–587.

Concept Summary

- The sum S_n of the first n terms of an arithmetic series is given by $S_n = \frac{n}{2}[2a_1 + (n-1)d]$ or $S_n = \frac{n}{2}(a_1 + a_n)$.

Example **Find S_n for the arithmetic series with $a_1 = 34$, $a_n = 2$, and $n = 9$.**

$S_n = \frac{n}{2}(a_1 + a_n)$ Sum formula

$S_9 = \frac{9}{2}(34 + 2)$ $n = 9, a_1 = 34, a_n = 2$

$S_9 = 162$ Simplify.

Exercises Find S_n for each arithmetic series. *See Examples on pages 584 and 585.*

17. $a_1 = 12, a_n = 117, n = 36$
18. $4 + 10 + 16 + \ldots + 106$
19. $10 + 4 + (-2) + \ldots + (-50)$
20. $\sum_{n=2}^{13} (3n + 1)$

11-3 Geometric Sequences

See pages 588–592.

Concept Summary

- A geometric sequence is one in which each term after the first is found by multiplying the previous term by a common ratio.
- The nth term a_n of a geometric sequence with first term a_1 and common ratio r is given by $a_n = a_1 \cdot r^{n-1}$, where n is any positive integer.

Examples

1 **Find the fifth term of a geometric sequence for which $a_1 = 7$ and $r = 3$.**

$a_n = a_1 \cdot r^{n-1}$ Formula for nth term

$a_5 = 7 \cdot 3^{5-1}$ $n = 5, a_1 = 7, r = 3$

$a_5 = 567$ The fifth term is 567.

2 **Find two geometric means between 1 and 8.**

$a_n = a_1 \cdot r^{n-1}$ Formula for nth term

$a_4 = 1 \cdot r^{4-1}$ $n = 4$ and $a_1 = 1$

$8 = r^3$ $a_4 = 8$

$2 = r$ The geometric means are 1(2) or 2 and 2(2) or 4.

Exercises **Find the indicated term of each geometric sequence.**
See Example 2 on page 589.

21. $a_1 = 2, r = 2, n = 5$

22. $a_1 = 7, r = 2, n = 4$

23. $a_1 = 243, r = -\frac{1}{3}, n = 5$

24. a_6 for $\frac{2}{3}, \frac{4}{3}, \frac{8}{3}, \ldots$

Find the geometric means in each sequence. *See Example 5 on page 590.*

25. 3, _?_, _?_, 24

26. 7.5, _?_, _?_, _?_, 120

27. 8, _?_, _?_, _?_, _?_, $\frac{1}{4}$

28. 5, _?_, _?_, _?_, 80

11-4 Geometric Series

See pages 594–598.

Concept Summary

- The sum S_n of the first n terms of a geometric series is given by

$$S_n = \frac{a_1(1 - r^n)}{1 - r} \text{ or } S_n = \frac{a_1 - a_1r^n}{1 - r}, \text{ where } r \neq 1.$$

Example **Find the sum of a geometric series for which $a_1 = 7$, $r = 3$, and $n = 14$.**

$S_n = \frac{a_1 - a_1r^n}{1 - r}$ Sum formula

$S_{14} = \frac{7 - 7 \cdot 3^{14}}{1 - 3}$ $n = 14, a_1 = 7, r = 3$

$S_{14} = 16{,}740{,}388$ Use a calculator.

Exercises **Find S_n for each geometric series.** *See Examples 1 and 3 on pages 595 and 596.*

29. $a_1 = 12, r = 3, n = 5$

30. $4 - 2 + 1 - \ldots$ to 6 terms

31. $256 + 192 + 144 + \ldots$ to 7 terms

32. $\sum_{n=1}^{5} \left(-\frac{1}{2}\right)^{n-1}$

11-5 Infinite Geometric Series

See pages 599–604.

Concept Summary

- The sum S of an infinite geometric series with $-1 < r < 1$ is given by $S = \frac{a_1}{1 - r}$.

Example **Find the sum of the infinite geometric series for which $a_1 = 18$ and $r = -\frac{2}{7}$.**

$S = \frac{a_1}{1 - r}$ Sum formula

$= \frac{18}{1 - \left(-\frac{2}{7}\right)}$ $a_1 = 18, r = -\frac{2}{7}$

$= \frac{18}{\frac{9}{7}}$ or 14 Simplify.

Exercises **Find the sum of each infinite geometric series, if it exists.**
See Example 1 on page 600.

33. $a_1 = 6, r = \frac{11}{12}$ **34.** $\frac{1}{8} - \frac{3}{16} + \frac{9}{32} - \frac{27}{64} + \ldots$ **35.** $\sum_{n=1}^{\infty} -2\left(-\frac{5}{8}\right)^{n-1}$

11-6 Recursion and Special Sequences

See pages 606–610.

Concept Summary

- In a recursive formula, each term is formulated from one or more previous terms.
- Iteration is the process of composing a function with itself repeatedly.

Examples

1 **Find the first five terms of the sequence in which $a_1 = 2$ and $a_{n+1} = 2a_n - 1$.**

$a_{n+1} = 2a_n - 1$ Recursive formula

$a_{1+1} = 2a_1 - 1$ $n = 1$
$a_2 = 2(2) - 1$ or 3 $a_1 = 2$
$a_{2+1} = 2a_2 - 1$ $n = 2$
$a_3 = 2(3) - 1$ or 5 $a_2 = 3$

$a_{3+1} = 2a_3 - 1$ $n = 3$
$a_4 = 2(5) - 1$ or 9 $a_3 = 5$
$a_{4+1} = 2a_4 - 1$ $n = 4$
$a_5 = 2(9) - 1$ or 17 $a_4 = 9$

The first five terms of the sequence are 2, 3, 5, 9, and 17.

2 **Find the first three iterates of $f(x) = -5x - 1$ for an initial value of $x_0 = -1$.**

$x_1 = f(x_0)$
$= f(-1)$
$= -5(-1) - 1$ or 4

$x_2 = f(x_1)$
$= f(4)$
$= -5(4) - 1$ or -21

$x_3 = f(x_2)$
$= f(-21)$
$= -5(-21) - 1$ or 104

The first three iterates are 4, −21, and 104.

Exercises **Find the first five terms of each sequence.** *See Example 1 on page 606.*

36. $a_1 = -2, a_{n+1} = a_n + 5$
37. $a_1 = 3, a_{n+1} = 4a_n - 10$
38. $a_1 = 2, a_{n+1} = a_n + 3n$
39. $a_1 = 1, a_2 = 3, a_{n+2} = a_{n+1} + a_n$

Find the first three iterates of each function for the given initial value.
See Example 3 on page 608.

40. $f(x) = -2x + 3, x_0 = 1$
41. $f(x) = 7x - 4, x_0 = 2$
42. $f(x) = x^2 - 6, x_0 = -1$
43. $f(x) = -2x^2 - x + 5, x_0 = -2$

11-7 The Binomial Theorem

See pages 612–617.

Concept Summary

- Pascal's triangle can be used to find the coefficients in a binomial expansion.
- The Binomial Theorem: $(a + b)^n = \sum_{k=0}^{n} \frac{n!}{(n-k)!k!} a^{n-k} b^k$

Chapter 11 For More ...
• Extra Practice, see pages 851–855.
• Mixed Problem Solving, see page 872.

Example **Expand $(a-2b)^4$.**

$$(a-2b)^4 = \sum_{k=0}^{4}\frac{4!}{(4-k)!k!}a^{4-k}(-2b)^k \quad \text{Binomial Theorem}$$

$$= \frac{4!}{4!0!}a^4(-2b)^0 + \frac{4!}{3!1!}a^3(-2b)^1 + \frac{4!}{2!2!}a^2(-2b)^2 + \frac{4!}{1!3!}a^1(-2b)^3 + \frac{4!}{0!4!}a^0(-2b)^4$$

$$= a^4 - 8a^3b + 24a^2b^2 - 32ab^3 + 16b^4 \quad \text{Simplify.}$$

Exercises **Expand each power.** *See Examples 1, 2, and 4 on pages 613 and 614.*

44. $(x+y)^3$ **45.** $(x-2)^4$ **46.** $(3r+s)^5$

Find the indicated term of each expansion. *See Example 5 on page 615.*

47. fourth term of $(x+2y)^6$ **48.** second term of $(4x-5)^{10}$

11-8 Proof and Mathematical Induction

See pages 618–621.

Concept Summary

- Mathematical induction is a method of proof used to prove statements about the positive integers.

Example **Prove $1 + 5 + 25 + \ldots + 5^{n-1} = \frac{1}{4}(5^n - 1)$ for all positive integers n.**

Step 1 When $n = 1$, the left side of the given equation is 1. The right side is $\frac{1}{4}(5^1 - 1)$ or 1. Thus, the equation is true for $n = 1$.

Step 2 Assume that $1 + 5 + 25 + \ldots + 5^{k-1} = \frac{1}{4}(5^k - 1)$ for some positive integer k.

Step 3 Show that the given equation is true for $n = k + 1$.

$$1 + 5 + 25 + \ldots + 5^{k-1} + 5^{(k+1)-1} = \frac{1}{4}(5^k - 1) + 5^{(k+1)-1} \quad \text{Add } 5^{(k+1)-1} \text{ to each side.}$$

$$= \frac{1}{4}(5^k - 1) + 5^k \quad \text{Simplify the exponent.}$$

$$= \frac{5^k - 1 + 4 \cdot 5^k}{4} \quad \text{Common denominator}$$

$$= \frac{5 \cdot 5^k - 1}{4} \quad \text{Distributive Property}$$

$$= \frac{1}{4}(5^{k+1} - 1) \quad 5 \cdot 5^k = 5^{k+1}$$

The last expression above is the right side of the equation to be proved, where n has been replaced by $k + 1$. Thus, the equation is true for $n = k + 1$.

This proves that $1 + 5 + 25 + \ldots + 5^{n-1} = \frac{1}{4}(5^n - 1)$ for all positive integers n.

Exercises **Prove that each statement is true for all positive integers.** *See Examples 1 and 2 on pages 618 and 619.*

49. $1 + 2 + 4 + \ldots + 2^{n-1} = 2^n - 1$ **50.** $6^n - 1$ is divisible by 5.

Chapter 11 Practice Test

Vocabulary and Concepts

Choose the correct term to complete each sentence.

1. A sequence in which each term after the first is found by adding a constant to the previous term is called a(n) (*arithmetic*, *geometric*) sequence.
2. A (*Fibonacci sequence*, *series*) is a sum of terms of a sequence.
3. (*Pascal's triangle*, *Recursive formulas*) and the Binomial Theorem can be used to expand powers of binomials.

Skills and Applications

4. Find the next four terms of the arithmetic sequence 42, 37, 32, … .
5. Find the 27th term of an arithmetic sequence for which $a_1 = 2$ and $d = 6$.
6. Find the three arithmetic means between −4 and 16.
7. Find the sum of the arithmetic series for which $a_1 = 7$, $n = 31$, and $a_n = 127$.
8. Find the next two terms of the geometric sequence $\frac{1}{81}, \frac{1}{27}, \frac{1}{9}, \ldots$.
9. Find the sixth term of the geometric sequence for which $a_1 = 5$ and $r = -2$.
10. Find the two geometric means between 7 and 189.
11. Find the sum of the geometric series for which $a_1 = 125$, $r = \frac{2}{5}$, and $n = 4$.

Find the sum of each series, if it exists.

12. $\sum_{k=3}^{15}(14 - 2k)$
13. $\sum_{n=1}^{\infty}\frac{1}{3}(-2)^{n-1}$
14. $91 + 85 + 79 + \ldots + (-29)$
15. $12 + (-6) + 3 - \frac{3}{2} + \ldots$

Find the first five terms of each sequence.

16. $a_1 = 1, a_{n+1} = a_n + 3$
17. $a_1 = -3, a_{n+1} = a_n + n^2$
18. Find the first three iterates of $f(x) = x^2 - 3x$ for an initial value of $x_0 = 1$.
19. Expand $(2s - 3t)^5$.
20. Find the third term of the expansion of $(x + y)^{10}$.

Prove that each statement is true for all positive integers.

21. $1 + 3 + 5 + \ldots + (2n - 1) = n^2$
22. $14^n - 1$ is divisible by 13.
23. **DESIGN** A landscaper is designing a wall of white brick and red brick. The pattern starts with 20 red bricks on the bottom row. Each row above it contains 3 fewer red bricks than the preceding row. If the top row contains no red bricks, how many rows are there and how many red bricks were used?
24. **RECREATION** One minute after it is released, a gas-filled balloon has risen 100 feet. In each succeeding minute, the balloon rises only 50% as far as it rose in the previous minute. How far will the balloon rise in 5 minutes?
25. **STANDARDIZED TEST PRACTICE** Find the next term in the geometric sequence $8, 6, \frac{9}{2}, \frac{27}{8}, \ldots$.

   (A) $\frac{11}{8}$ (B) $\frac{27}{16}$ (C) $\frac{9}{4}$ (D) $\frac{81}{32}$

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Chapter 11 Standardized Test Practice

Part 1 Multiple Choice

Record your answers on the answer sheet provided by your teacher or on a sheet of paper.

1. For all positive integers, let $\boxed{n} = n + g$, where g is the greatest factor of n, and $g < n$. If $\boxed{18} = x$, then $\boxed{x} =$

(A) 9. (B) 8.
(C) 27. (D) 36.

2. If p is positive, what percent of $6p$ is 12?

(A) $\frac{p}{100}\%$ (B) $\frac{p}{2}\%$
(C) $\frac{12}{p}\%$ (D) $\frac{200}{p}\%$

3. A box is 12 units tall, 6 units long, and 8 units wide. A designer is creating a new box that must have the same volume as the first box. If the length and width of the new box are each 50% greater than the length and width of the first box, about how many units tall will the new box be?

(A) 5.3 (B) 6.8
(C) 7.1 (D) 8.5

4. Which of the following statements must be true when $0 < m < 1$?

I $\frac{\sqrt{m}}{m} > 1$ **II** $4m < 1$ **III** $m^2 - m^3 < 0$

(A) I only
(B) III only
(C) I and II only
(D) I, II, and III

5. If $3kx - \frac{4s}{t} = 3ky$, then $x - y = ?$

(A) $-\frac{4s}{3kt}$ (B) $\frac{-4s}{t} + \frac{1}{3k}$
(C) $\frac{4s}{3t} - k$ (D) $\frac{4s}{3kt}$

6. For all $n \neq 0$, what is the slope of the line passing through $(3n, -k)$ and $(-n, -k)$?

(A) 0 (B) $\frac{k}{2n}$
(C) $\frac{2n}{k}$ (D) undefined

7. Which is the graph of the equation $x^2 + (y - 4)^2 = 20$?

(A) line (B) parabola
(C) circle (D) ellipse

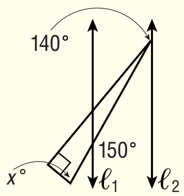
8. $\dfrac{x - \frac{9}{x}}{1 - \frac{6}{x} + \frac{9}{x^2}} =$

(A) $\frac{x}{x - 3}$ (B) $\frac{x^2 + 3}{x - 3}$
(C) $\frac{x^2 + 3x}{x - 3}$ (D) $\frac{x^2 + 3x}{(x - 3)^2}$

9. What is the sum of the positive even factors of 30?

(A) 18 (B) 30
(C) 48 (D) 72

10. If ℓ_1 is parallel to ℓ_2 in the figure, what is the value of x?



(A) 30
(B) 40
(C) 70
(D) 80

The Princeton Review Test-Taking Tip

Question 5 Some questions ask you to find the value of an expression. It is often not necessary to find the value of each variable in the expression. For example, to answer Question 5, it is not necessary to find the values of x and y. Isolate the expression $x - y$ on one side of the equation.

Aligned and verified by The Princeton Review

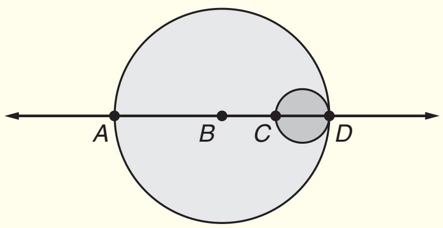
Part 2 Short Response/Grid In

Record your answers on the answer sheet provided by your teacher or on a sheet of paper.

11. $\begin{array}{r} AA \\ +\,BB \\ \hline CC \end{array}$

If A, B, and C are each digits and $A = 3B$, then what is one possible value of C?

12. In the figure, each arc is a semicircle. If B is the midpoint of $\overline{AD}$ and C is the midpoint of $\overline{BD}$, what is the ratio of the area of the semicircle $\overset{\frown}{CD}$ to the area of the semicircle $\overset{\frown}{AD}$?



13. Two people are 17.5 miles apart. They begin to walk toward each other along a straight line at the same time. One walks at the rate of 4 miles per hour, and the other walks at the rate of 3 miles per hour. In how many hours will they meet?

14. If $\frac{x + y}{x} = \frac{5}{4}$, then $\frac{y}{x} =$

15. A car's gasoline tank is $\frac{1}{2}$ full. After adding 7 gallons of gas, the gauge shows that the tank is $\frac{3}{4}$ full. How many gallons does the tank hold?

16. If $a = 15 - b$, what is the value of $3a + 3b$?

17. If $x^9 = \frac{45}{y}$ and $x^7 = \frac{1}{5y}$, and $x > 0$, what is the value of x?

Part 3 Quantitative Comparison

Compare the quantity in Column A and the quantity in Column B. Then determine whether:

- **(A) the quantity in Column A is greater,**
- **(B) the quantity in Column B is greater,**
- **(C) the two quantities are equal, or**
- **(D) the relationship cannot be determined from the information given.**

	Column A	Column B
18.	the arithmetic mean of three consecutive integers where x is the median	the arithmetic mean of five consecutive integers where x is the median

19. The area of Square B is equal to nine times the area of Square A.

	Column A	Column B
	three times the perimeter of Square A	the perimeter of Square B

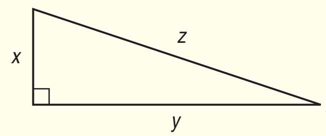
20. $\diamond n \diamond = n(n + 1)$ if n is even

$\diamond n \diamond = n(n - 1)$ if n is odd

	Column A	Column B
	$\diamond 8 \diamond$	$\diamond 9 \diamond$

	Column A	Column B
21.	$(1 - \sqrt{3})(1 - \sqrt{3})$	$(1 - \sqrt{3})(1 + \sqrt{3})$

22.



	Column A	Column B
	$\frac{x + y}{2}$	$\frac{x + z}{2}$

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