Chapter 10

Exponential and Logarithmic Relations

What You'll Learn

- **Lessons 10-1 through 10-3** Simplify exponential and logarithmic expressions.
- **Lessons 10-1, 10-4, and 10-5** Solve exponential equations and inequalities.
- **Lessons 10-2 and 10-3** Solve logarithmic equations and inequalities.
- **Lesson 10-6** Solve problems involving exponential growth and decay.

Key Vocabulary

- exponential growth (p. 524)
- exponential decay (p. 524)
- logarithm (p. 531)
- common logarithm (p. 547)
- natural logarithm (p. 554)

Why It's Important

Exponential functions are often used to model problems involving growth and decay. Logarithms can also be used to solve such problems. *You will learn how a declining farm population can be modeled by an exponential function in Lesson 10-1.*

Getting Started

Prerequisite Skills To be successful in this chapter, you'll need to master these skills and be able to apply them in problem-solving situations. Review these skills before beginning Chapter 10.

Lessons 10-1 through 10-3 — Multiply and Divide Monomials

Simplify. Assume that no variable equals 0. *(For review, see Lesson 5-1.)*

1. $x^5 \cdot x \cdot x^6$ **2.** $(3ab^4c^2)^3$ **3.** $\dfrac{-36x^7y^4z^3}{21x^4y^9z^4}$ **4.** $\left(\dfrac{4ab^2}{64b^3c}\right)^2$

Lessons 10-2 and 10-3 — Solve Inequalities

Solve each inequality. *(For review, see Lesson 1-5)*

5. $a + 4 < -10$ **6.** $-5n \leq 15$ **7.** $3y + 2 \geq -4$ **8.** $15 - x > 9$

Lessons 10-2 and 10-3 — Inverse Functions

Find the inverse of each function. Then graph the function and its inverse.
(For review, see Lesson 7-8.)

9. $f(x) = -2x$ **10.** $f(x) = 3x - 2$ **11.** $f(x) = -x + 1$ **12.** $f(x) = \dfrac{x - 4}{3}$

Lessons 10-2 and 10-3 — Composition of Functions

Find $g[h(x)]$ and $h[g(x)]$. *(For review, see Lesson 7-7.)*

13. $h(x) = 3x + 4$
$g(x) = x - 2$

14. $h(x) = 2x - 7$
$g(x) = 5x$

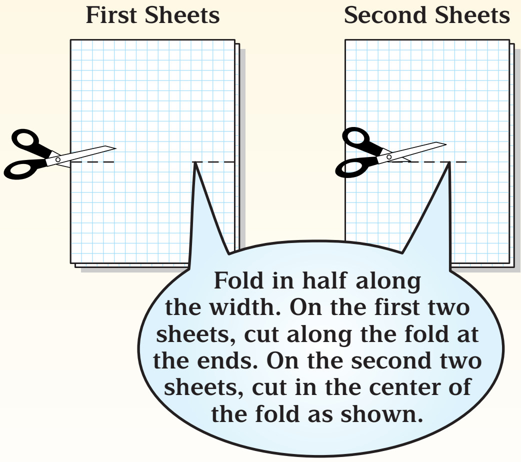
15. $h(x) = x - 4$
$g(x) = x^2$

16. $h(x) = 4x + 1$
$g(x) = -2x - 3$

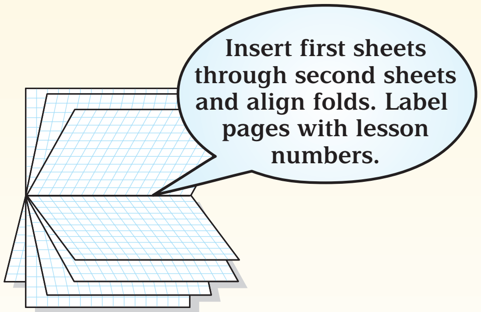
FOLDABLES™ Study Organizer

Make this Foldable to record information about exponential and logarithmic relations. Begin with four sheets of grid paper.

Step 1 Fold and Cut



Step 2 Fold and Label



Reading and Writing As you read and study the chapter, fill the journal with notes, diagrams, and examples for each lesson.

Investigating Exponential Functions

Collect the Data

Step 1 Cut a sheet of notebook paper in half.

Step 2 Stack the two halves, one on top of the other.

Step 3 Make a table like the one below and record the number of sheets of paper you have in the stack after one cut.

Number of Cuts	Number of Sheets
0	1
1	
2	

Step 4 Cut the two stacked sheets in half, placing the resulting pieces in a single stack. Record the number of sheets of paper in the new stack after 2 cuts.

Step 5 Continue cutting the stack in half, each time putting the resulting piles in a single stack and recording the number of sheets in the stack. Stop when the resulting stack is too thick to cut.

Analyze the Data

1. Write a list of ordered pairs (x, y), where x is the number of cuts and y is the number of sheets in the stack. Notice that the list starts with the ordered pair (0, 1), which represents the single sheet of paper before any cuts were made.
2. Continue the list, beyond the point where you stopped cutting, until you reach the ordered pair for 7 cuts. Explain how you calculated the last y values for your list, after you had stopped cutting.
3. Plot the ordered pairs in your list on a coordinate grid. Be sure to choose a scale for the y-axis so that you can plot all of the points.
4. Describe the pattern of the points you have plotted. Do they lie on a straight line?

Make a Conjecture

5. Write a function that expresses y as a function of x.
6. Use a calculator to evaluate the function you wrote in Exercise 5 for $x = 8$ and $x = 9$. Does it give the correct number of sheets in the stack after 8 and 9 cuts?
7. Notebook paper usually stacks about 500 sheets to the inch. How thick would your stack of paper be if you had been able to make 9 cuts?
8. Suppose each cut takes about 5 seconds. If you had been able to keep cutting, you would have made 36 cuts in three minutes. At 500 sheets to the inch, make a conjecture as to how thick you think the stack would be after 36 cuts.
9. Use your function from Exercise 5 to calculate the thickness of your stack after 36 cuts. Write your answer in miles.

10-1 Exponential Functions

What You'll Learn

- Graph exponential functions.
- Solve exponential equations and inequalities.

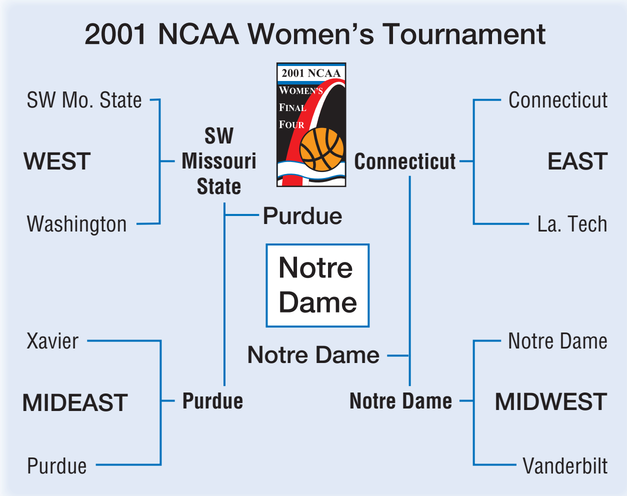
Vocabulary

- exponential function
- exponential growth
- exponential decay
- exponential equation
- exponential inequality

How does an exponential function describe tournament play?

The NCAA women's basketball tournament begins with 64 teams and consists of 6 rounds of play. The winners of the first round play against each other in the second round. The winners then move from the Sweet Sixteen to the Elite Eight to the Final Four and finally to the Championship Game.

The number of teams y that compete in a tournament of x rounds is $y = 2^x$.



Study Tip

Common Misconception
Be sure not to confuse polynomial functions and exponential functions. While $y = x^2$ and $y = 2^x$ each have an exponent, $y = x^2$ is a polynomial function and $y = 2^x$ is an exponential function.

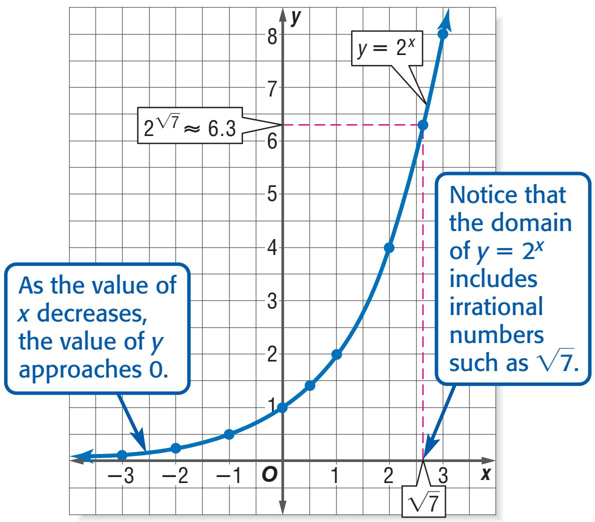
EXPONENTIAL FUNCTIONS In an exponential function like $y = 2^x$, the base is a constant, and the exponent is a variable. Let's examine the graph of $y = 2^x$.

Example 1 Graph an Exponential Function

Sketch the graph of $y = 2^x$. Then state the function's domain and range.

Make a table of values. Connect the points to sketch a smooth curve.

x	$y = 2^x$
-3	$2^{-3} = \frac{1}{8}$
-2	$2^{-2} = \frac{1}{4}$
-1	$2^{-1} = \frac{1}{2}$
0	$2^0 = 1$
$\frac{1}{2}$	$2^{\frac{1}{2}} = \sqrt{2}$
1	$2^1 = 2$
2	$2^2 = 4$
3	$2^3 = 8$



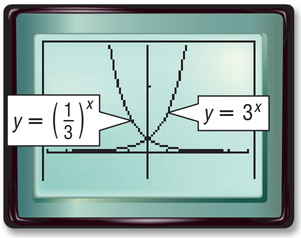
The domain is all real numbers, while the range is all positive numbers.

You can use a TI-83 Plus graphing calculator to look at the graph of two other exponential functions, $y = 3x$ and $y = \left(\frac{1}{3}\right)^x$.

Graphing Calculator Investigation

Families of Exponential Functions

The calculator screen shows the graphs of $y = 3^x$ and $y = \left(\frac{1}{3}\right)^x$.



[−5, 5] scl: 1 by [−2, 8] scl: 1

Think and Discuss

1. How do the shapes of the graphs compare?
2. How do the asymptotes and y-intercepts of the graphs compare?
3. Describe the relationship between the graphs.
4. Graph each group of functions on the same screen. Then compare the graphs, listing both similarities and differences in shape, asymptotes, domain, range, and y-intercepts.
   a. $y = 2^x$, $y = 3^x$, and $y = 4^x$
   b. $y = \left(\frac{1}{2}\right)^x$, $y = \left(\frac{1}{3}\right)^x$, and $y = \left(\frac{1}{4}\right)^x$
   c. $y = -3(2)^x$ and $y = 3(2)^x$; $y = -1(2)^x$ and $y = 2^x$.
5. Describe the relationship between the graphs of $y = -1(2)^x$ and $y = 2^x$.

Study Tip

Look Back

To review **continuous functions**, see page 63, Exercises 60 and 61. To review **one-to-one functions**, see Lesson 2-1.

In general, an equation of the form $y = ab^x$, where $a \neq 0$, $b > 0$, and $b \neq 1$, is called an **exponential function** with base b. Exponential functions have the following characteristics.

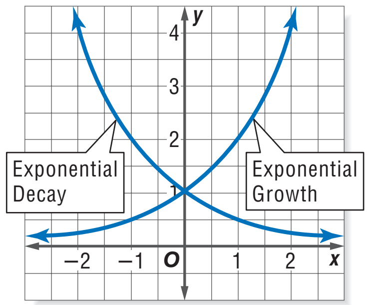
1. The function is continuous and one-to-one.
2. The domain is the set of all real numbers.
3. The x-axis is an asymptote of the graph.
4. The range is the set of all positive numbers if $a > 0$ and all negative numbers if $a < 0$.
5. The graph contains the point $(0, a)$. That is, the y-intercept is a.
6. The graphs of $y = ab^x$ and $y = a\left(\frac{1}{b}\right)^x$ are reflections across the y-axis.

Study Tip

Exponential Growth and Decay

Notice that the graph of an exponential growth function *rises* from left to right. The graph of an exponential decay function *falls* from left to right.

There are two types of exponential functions: **exponential growth** and **exponential decay**. The base of an exponential growth function is a number greater than one. The base of an exponential decay function is a number between 0 and 1.



Key Concept — *Exponential Growth and Decay*

- If $a > 0$ and $b > 1$, the function $y = ab^x$ represents exponential growth.
- If $a > 0$ and $0 < b < 1$, the function $y = ab^x$ represents exponential decay.

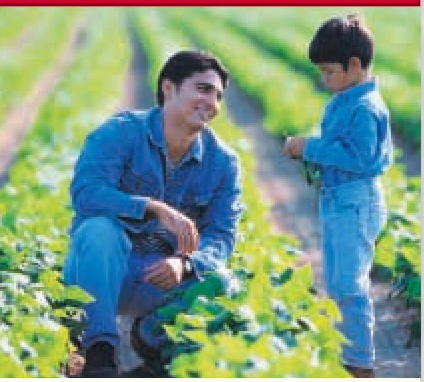
Example 2 Identify Exponential Growth and Decay

Determine whether each function represents exponential *growth* or *decay*.

Function	Exponential Growth or Decay?
a. $y = \left(\frac{1}{5}\right)^x$	The function represents exponential decay, since the base, $\frac{1}{5}$, is between 0 and 1.
b. $y = 3(4)^x$	The function represents exponential growth, since the base, 4, is greater than 1.
c. $y = 7(1.2)^x$	The function represents exponential growth, since the base, 1.2, is greater than 1.

Exponential functions are frequently used to model the growth or decay of a population. You can use the y-intercept and one other point on the graph to write the equation of an exponential function.

More About. . .



Farming

In 1999, 47% of the net farm income in the United States was from direct government payments. The USDA has set a goal of reducing this percent to 14% by 2005.

Source: USDA

Example 3 Write an Exponential Function

FARMING **In 1983, there were 102,000 farms in Minnesota, but by 1998, this number had dropped to 80,000.**

a. Write an exponential function of the form $y = ab^x$ that could be used to model the farm population y of Minnesota. Write the function in terms of x, the number of years since 1983.

For 1983, the time x equals 0, and the initial population y is 102,000. Thus, the y-intercept, and value of a, is 102,000.

For 1998, the time x equals 1998 − 1983 or 15, and the population y is 80,000. Substitute these values and the value of a into an exponential function to approximate the value of b.

$y = ab^x$	Exponential function
$80{,}000 = 102{,}000b^{15}$	Replace x with 15, y with 80,000, and a with 102,000.
$0.78 \approx b^{15}$	Divide each side by 102,000.
$\sqrt[15]{0.78} \approx b$	Take the 15th root of each side.

To find the 15th root of 0.78, use selection 5: $\sqrt[x]{}$ under the MATH menu on the TI-83 Plus.

KEYSTROKES: 15 [MATH] 5 0.78 [ENTER] .9835723396

An equation that models the farm population of Minnesota from 1983 to 1998 is $y = 102{,}000(0.98)^x$.

b. Suppose the number of farms in Minnesota continues to decline at the same rate. Estimate the number of farms in 2010.

For 2010, the time x equals 2010 − 1983 or 27.

$y = 102{,}000(0.98)^x$	Modeling equation
$y = 102{,}000(0.98)^{27}$	Replace x with 27.
$y \approx 59{,}115$	Use a calculator.

The farm population in Minnesota will be about 59,115 in 2010.

Study Tip

Look Back

To review **Properties of Power**, see Lesson 5-1.

EXPONENTIAL EQUATIONS AND INEQUALITIES Since the domain of an exponential function includes irrational numbers such as $\sqrt{2}$, all the properties of rational exponents apply to irrational exponents.

Example 4 Simplify Expressions with Irrational Exponents

Simplify each expression.

a. $2^{\sqrt{5}} \cdot 2^{\sqrt{3}}$

$2^{\sqrt{5}} \cdot 2^{\sqrt{3}} = 2^{\sqrt{5} + \sqrt{3}}$ Product of Powers

b. $\left(7^{\sqrt{2}}\right)^{\sqrt{3}}$

$\left(7^{\sqrt{2}}\right)^{\sqrt{3}} = 7^{\sqrt{2} \cdot \sqrt{3}}$ Power of a Power

$= 7^{\sqrt{6}}$ Product of Radicals

The following property is useful for solving exponential equations. **Exponential equations** are equations in which variables occur as exponents.

Key Concept — Property of Equality for Exponential Functions

- **Symbols** If b is a positive number other than 1, then $b^x = b^y$ if and only if $x = y$.
- **Example** If $2^x = 2^8$, then $x = 8$.

Example 5 Solve Exponential Equations

Solve each equation.

a. $3^{2n+1} = 81$

$3^{2n+1} = 81$	Original equation
$3^{2n+1} = 3^4$	Rewrite 81 as 3^4 so each side has the same base.
$2n + 1 = 4$	Property of Equality for Exponential Functions
$2n = 3$	Subtract 1 from each side.
$n = \frac{3}{2}$	Divide each side by 2.

The solution is $\frac{3}{2}$.

CHECK

$3^{2n+1} = 81$	Original equation
$3^{2\left(\frac{3}{2}\right)+1} \stackrel{?}{=} 81$	Substitute $\frac{3}{2}$ for n.
$3^4 \stackrel{?}{=} 81$	Simplify.
$81 = 81$ ✓	Simplify.

b. $4^{2x} = 8^{x-1}$

$4^{2x} = 8^{x-1}$	Original equation
$(2^2)^{2x} = (2^3)^{x-1}$	Rewrite each side with a base of 2.
$2^{4x} = 2^{3(x-1)}$	Power of a Power
$4x = 3(x - 1)$	Property of Equality for Exponential Functions
$4x = 3x - 3$	Distributive Property
$x = -3$	Subtract $3x$ from each side.

The solution is -3.

The following property is useful for solving inequalities involving exponential functions or **exponential inequalities**.

Key Concept *Property of Inequality for Exponential Functions*

- **Symbols** If $b > 1$, then $b^x > b^y$ if and only if $x > y$, and $b^x < b^y$ if and only if $x < y$.
- **Example** If $5^x < 5^4$, then $x < 4$.

This property also holds for $\le$ and $\ge$.

Example 6 Solve Exponential Inequalities

Solve $4^{3p-1} > \frac{1}{256}$.

$4^{3p-1} > \frac{1}{256}$ Original inequality

$4^{3p-1} > 4^{-4}$ Rewrite $\frac{1}{256}$ as $\frac{1}{4^4}$ or 4^{-4} so each side has the same base.

$3p - 1 > -4$ Property of Inequality for Exponential Functions

$3p > -3$ Add 1 to each side.

$p > -1$ Divide each side by 3.

The solution set is $p > -1$.

CHECK Test a value of p greater than -1; for example, $p = 0$.

$4^{3p-1} > \frac{1}{256}$ Original inequality

$4^{3(0)-1} \stackrel{?}{>} \frac{1}{256}$ Replace p with 0.

$4^{-1} \stackrel{?}{>} \frac{1}{256}$ Simplify.

$\frac{1}{4} > \frac{1}{256}$ ✓ $a^{-1} = \frac{1}{a}$

Check for Understanding

Concept Check

1. **OPEN ENDED** Give an example of a value of b for which $y = b^x$ represents exponential decay.

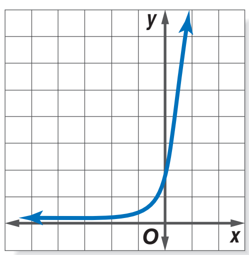
2. **Identify** each function as *linear*, *quadratic*, or *exponential*.

   a. $y = 3x^2$ b. $y = 4(3)^x$ c. $y = 2x + 4$ d. $y = 4(0.2)^x + 1$

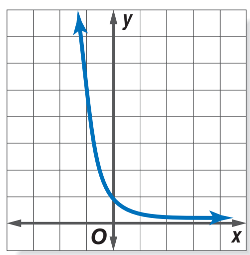
Match each function with its graph.

3. $y = 5^x$ 4. $y = 2(5)^x$ 5. $y = \left(\frac{1}{5}\right)^x$

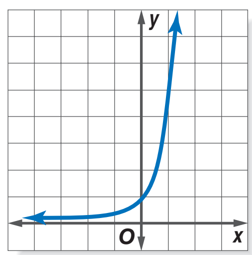
a.



b.



c.



Guided Practice

Sketch the graph of each function. Then state the function's domain and range.

6. $y = 3(4)^x$ 7. $y = 2\left(\frac{1}{3}\right)^x$

Determine whether each function represents exponential *growth* or *decay*.

8. $y = 2(7)^x$ **9.** $y = (0.5)^x$ **10.** $y = 0.3(5)^x$

Write an exponential function whose graph passes through the given points.

11. (0, 3) and (−1, 6) **12.** (0, −18) and (−2, −2)

Simplify each expression.

13. $2^{\sqrt{7}} \cdot 2^{\sqrt{7}}$ **14.** $(a^{\pi})^4$ **15.** $81^{\sqrt{2}} \div 3^{\sqrt{2}}$

Solve each equation or inequality. Check your solution.

16. $2^{n+4} = \frac{1}{32}$ **17.** $5^{2x+3} \leq 125$ **18.** $9^{2y-1} = 27^y$

Application

ANIMAL CONTROL **For Exercises 19 and 20, use the following information.**
During the 19th century, rabbits were brought to Australia. Since the rabbits had no natural enemies on that continent, their population increased rapidly. Suppose there were 65,000 rabbits in Australia in 1865 and 2,500,000 in 1867.

19. Write an exponential function that could be used to model the rabbit population y in Australia. Write the function in terms of x, the number of years since 1865.

20. Assume that the rabbit population continued to grow at that rate. Estimate the Australian rabbit population in 1872.

Practice and Apply

Homework Help

For Exercises	See Examples
21–26	1
27–32	2
33–38, 57–66	3
39–44	4
45–56	5, 6

Extra Practice

See page 849.

Sketch the graph of each function. Then state the function's domain and range.

21. $y = 2(3)^x$ **22.** $y = 5(2)^x$ **23.** $y = 0.5(4)^x$

24. $y = 4\left(\frac{1}{3}\right)^x$ **25.** $y = -\left(\frac{1}{5}\right)^x$ **26.** $y = -2.5(5)^x$

Determine whether each function represents exponential *growth* or *decay*.

27. $y = 10(3.5)^x$ **28.** $y = 2(4)^x$ **29.** $y = 0.4\left(\frac{1}{3}\right)^x$

30. $y = 3\left(\frac{5}{2}\right)^x$ **31.** $y = 30^{-x}$ **32.** $y = 0.2(5)^{-x}$

Write an exponential function whose graph passes through the given points.

33. (0, −2) and (−2, −32) **34.** (0, 3) and (1, 15)

35. (0, 7) and (2, 63) **36.** (0, −5) and (−3, −135)

37. (0, 0.2) and (4, 51.2) **38.** (0, −0.3) and (5, −9.6)

Simplify each expression.

39. $\left(5^{\sqrt{2}}\right)^{\sqrt{8}}$ **40.** $\left(x^{\sqrt{5}}\right)^{\sqrt{3}}$ **41.** $7^{\sqrt{2}} \cdot 7^{3\sqrt{2}}$

42. $y^{3\sqrt{3}} \div y^{\sqrt{3}}$ **43.** $n^2 \cdot n^{\pi}$ **44.** $64^{\pi} \div 2^{\pi}$

Solve each equation or inequality. Check your solution.

45. $3^{n-2} = 27$ **46.** $2^{3x+5} = 128$ **47.** $5^{n-3} = \frac{1}{25}$

48. $2^{2n} \leq \frac{1}{16}$ **49.** $\left(\frac{1}{9}\right)^m = 81^{m+4}$ **50.** $\left(\frac{1}{7}\right)^{y-3} = 343$

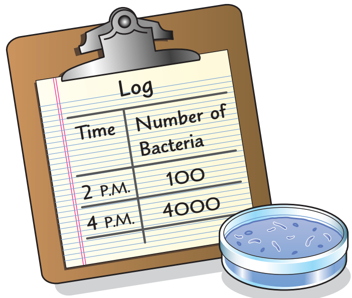
51. $16^n < 8^{n+1}$ **52.** $10^{x-1} = 100^{2x-3}$ **53.** $36^{2p} = 216^{p-1}$

54. $32^{5p+2} \geq 16^{5p}$ **55.** $3^{5x} \cdot 81^{1-x} = 9^{x-3}$ **56.** $49^x = 7^{x^2-15}$

WebQuest

The magnitude of an earthquake can be represented by an exponential equation. Visit www.algebra2.com/webquest to continue work on your WebQuest project.

BIOLOGY **For Exercises 57 and 58, use the following information.**
The number of bacteria in a colony is growing exponentially.



57. Write an exponential function to model the population y of bacteria x hours after 2 P.M.

58. How many bacteria were there at 7 P.M. that day?

POPULATION **For Exercises 59–61, use the following information.**
Every ten years, the Bureau of the Census counts the number of people living in the United States. In 1790, the population of the U.S. was 3.93 million. By 1800, this number had grown to 5.31 million.

59. Write an exponential function that could be used to model the U.S. population y in millions for 1790 to 1800. Write the equation in terms of x, the number of decades x since 1790.

60. Assume that the U.S. population continued to grow at that rate. Estimate the population for the years 1820, 1840, and 1860. Then compare your estimates with the actual population for those years, which were 9.64, 17.06, and 31.44 million, respectively.

61. RESEARCH Estimate the population of the U.S. in 2000. Then use the Internet or other reference to find the actual population of the U.S. in 2000. Has the population of the U.S. continued to grow at the same rate at which it was growing in the early 1800s? Explain.

More About. . .

Computers

Since computers were invented, computational speed has multiplied by a factor of 4 about every three years.

Source: www.wired.com

MONEY **For Exercises 62–64, use the following information.**
Suppose you deposit a principal amount of P dollars in a bank account that pays compound interest. If the annual interest rate is r (expressed as a decimal) and the bank makes interest payments n times every year, the amount of money A you would have after t years is given by $A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$.

62. If the principal, interest rate, and number of interest payments are known, what type of function is $A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$? Explain your reasoning.

63. Write an equation giving the amount of money you would have after t years if you deposit \$1000 into an account paying 4% annual interest compounded quarterly (four times per year).

64. Find the account balance after 20 years.

COMPUTERS **For Exercises 65 and 66, use the information at the left.**

65. If a typical computer operates with a computational speed s today, write an expression for the speed at which you can expect an equivalent computer to operate after x three-year periods.

66. Suppose your computer operates with a processor speed of 600 megahertz and you want a computer that can operate at 4800 megahertz. If a computer with that speed is currently unavailable for home use, how long can you expect to wait until you can buy such a computer?

67. CRITICAL THINKING Decide whether the following statement is *sometimes*, *always*, or *never* true. Explain your reasoning.

For a positive base b other than 1, $b^x > b^y$ if and only if $x > y$.

68. **WRITING IN MATH** Answer the question that was posed at the beginning of the lesson.

How does an exponential function describe tournament play?

Include the following in your answer:

- an explanation of how you could use the equation $y = 2^x$ to determine the number of rounds of tournament play for 128 teams, and
- an example of an inappropriate number of teams for tournament play with an explanation as to why this number would be inappropriate.

Standardized Test Practice

69. If $4^{x+2} = 48$, then $4^x =$

(A) 3.0. (B) 6.4. (C) 6.9. (D) 12.0. (E) 24.0.

70. **GRID IN** Suppose you deposit $500 in an account paying 4.5% interest compounded semiannually. Find the dollar value of the account rounded to the nearest penny after 10 years.

Graphing Calculator

FAMILIES OF GRAPHS **Graph each pair of functions on the same screen. Then compare the graphs, listing both similarities and differences in shape, asymptotes, domain, range, and y-intercepts.**

71. $y = 2^x$ and $y = 2^x + 3$

72. $y = 3^x$ and $y = 3^{x+1}$

73. $y = \left(\frac{1}{5}\right)^x$ and $y = \left(\frac{1}{5}\right)^{x-2}$

74. $y = \left(\frac{1}{4}\right)^x$ and $y = \left(\frac{1}{4}\right)^x - 1$

75. Describe the effect of changing the values of h and k in the equation $y = 2^{x-h} + k$.

Maintain Your Skills

Mixed Review

Solve each equation or inequality. Check your solutions. *(Lesson 9-6)*

76. $\frac{15}{p} + p = 16$

77. $\frac{s-3}{s+4} = \frac{6}{s^2-16}$

78. $\frac{2a-5}{a-9} + \frac{a}{a+9} = \frac{-6}{a^2-81}$

79. $\frac{x-2}{x} < \frac{x-4}{x-6}$

Identify each equation as a type of function. Then graph the equation. *(Lesson 9-5)*

80. $y = \sqrt{x-2}$

81. $y = -2[\![x]\!]$

82. $y = 8$

Find the inverse of each matrix, if it exists. *(Lesson 4-7)*

83. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

84. $\begin{bmatrix} 2 & 4 \\ 5 & 10 \end{bmatrix}$

85. $\begin{bmatrix} -5 & 6 \\ -11 & 3 \end{bmatrix}$

86. **ENERGY** A circular cell must deliver 18 watts of energy. If each square centimeter of the cell that is in sunlight produces 0.01 watt of energy, how long must the radius of the cell be? *(Lesson 5-8)*

Getting Ready for the Next Lesson

PREREQUISITE SKILL **Find $g[h(x)]$ and $h[g(x)]$.**

*(To review **composition of functions**, see Lesson 7-7.)*

87. $h(x) = 2x - 1$
$g(x) = x - 5$

88. $h(x) = x + 3$
$g(x) = x^2$

89. $h(x) = 2x + 5$
$g(x) = -x + 3$

10-2 Logarithms and Logarithmic Functions

What You'll Learn

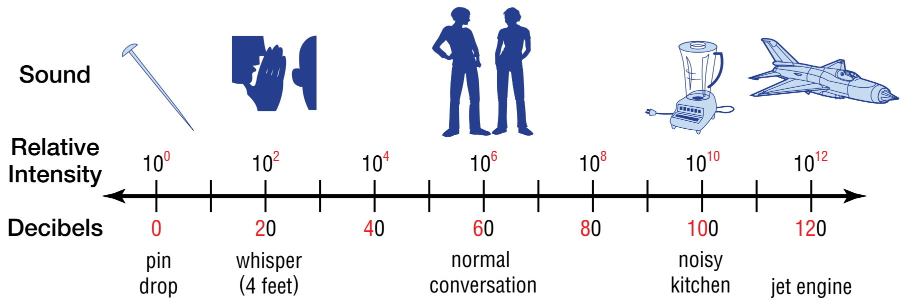
- Evaluate logarithmic expressions.
- Solve logarithmic equations and inequalities.

Vocabulary

- logarithm
- logarithmic function
- logarithmic equation
- logarithmic inequality

Why is a logarithmic scale used to measure sound?

Many scientific measurements have such an enormous range of possible values that it makes sense to write them as powers of 10 and simply keep track of their exponents. For example, the loudness of sound is measured in units called *decibels*. The graph shows the relative intensities and decibel measures of common sounds.



The decibel measure of the loudness of a sound is the exponent or logarithm of its relative intensity multiplied by 10.

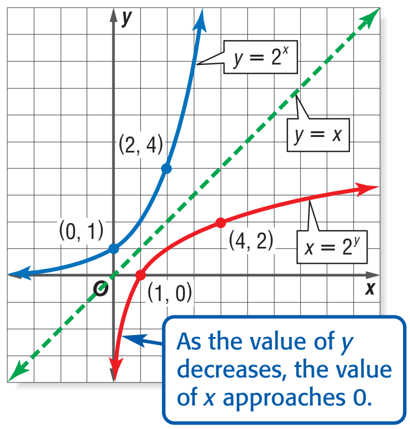
LOGARITHMIC FUNCTIONS AND EXPRESSIONS To better understand what is meant by a logarithm, let's look at the graph of $y = 2^x$ and its inverse. Since exponential functions are one-to-one, the inverse of $y = 2^x$ exists and is also a function. Recall that you can graph the inverse of a function by interchanging the x and y values in the ordered pairs of the function.

Study Tip

Look Back
To review **inverse functions**, see Lesson 7-8.

$y = 2^x$	
x	y
-3	$\frac{1}{8}$
-2	$\frac{1}{4}$
-1	$\frac{1}{2}$
0	1
1	2
2	4
3	8

$x = 2^y$	
x	y
$\frac{1}{8}$	-3
$\frac{1}{4}$	-2
$\frac{1}{2}$	-1
1	0
2	1
4	2
8	3



The inverse of $y = 2^x$ can be defined as $x = 2^y$. Notice that the graphs of these two functions are reflections of each other over the line $y = x$.

In general, the inverse of $y = b^x$ is $x = b^y$. In $x = b^y$, y is called the **logarithm** of x. It is usually written as $y = \log_b x$ and is read *y equals log base b of x*.

Key Concept — Logarithm with Base b

- **Words** Let b and x be positive numbers, $b \neq 1$. The *logarithm of x with base b* is denoted $\log_b x$ and is defined as the exponent y that makes the equation $b^y = x$ true.
- **Symbols** Suppose $b > 0$ and $b \neq 1$. For $x > 0$, there is a number y such that $\log_b x = y$ if and only if $b^y = x$.

Example 1 Logarithmic to Exponential Form

Write each equation in exponential form.

a. $\log_8 1 = 0$

$\log_8 1 = 0 \rightarrow 1 = 8^0$

b. $\log_2 \frac{1}{16} = -4$

$\log_2 \frac{1}{16} = -4 \rightarrow \frac{1}{16} = 2^{-4}$

Example 2 Exponential to Logarithmic Form

Write each equation in logarithmic form.

a. $10^3 = 1000$

$10^3 = 1000 \rightarrow \log_{10} 1000 = 3$

b. $9^{\frac{1}{2}} = 3$

$9^{\frac{1}{2}} = 3 \rightarrow \log_9 3 = \frac{1}{2}$

You can use the definition of logarithm to find the value of a logarithmic expression.

Example 3 Evaluate Logarithmic Expressions

Evaluate $\log_2 64$.

$\log_2 64 = y$	Let the logarithm equal y.
$64 = 2^y$	Definition of logarithm
$2^6 = 2^y$	$64 = 2^6$
$6 = y$	Property of Equality for Exponential Functions

So, $\log_2 64 = 6$.

The function $y = \log_b x$, where $b > 0$ and $b \neq 1$, is called a **logarithmic function**. As shown in the graph on the previous page, this function is the inverse of the exponential function $y = b^x$ and has the following characteristics.

1. The function is continuous and one-to-one.
2. The domain is the set of all positive real numbers.
3. The y-axis is an asymptote of the graph.
4. The range is the set of all real numbers.
5. The graph contains the point $(1, 0)$. That is, the x-intercept is 1.

Study Tip

Look Back

To review **composition of functions**, see Lesson 7-7.

Since the exponential function $f(x) = b^x$ and the logarithmic function $g(x) = \log_b x$ are inverses of each other, their composites are the identity function. That is, $f[g(x)] = x$ and $g[f(x)] = x$.

$$f[g(x)] = x \qquad g[f(x)] = x$$
$$f(\log_b x) = x \qquad g(b^x) = x$$
$$b^{\log_b x} = x \qquad \log_b b^x = x$$

Thus, if their bases are the same, exponential and logarithmic functions "undo" each other. You can use this inverse property of exponents and logarithms to simplify expressions.

Example 4 Inverse Property of Exponents and Logarithms

Evaluate each expression.

a. $\log_6 6^8$

$\log_6 6^8 = 8$ $\quad \log_b b^x = x$

b. $3^{\log_3 (4x - 1)}$

$3^{\log_3 (4x-1)} = 4x - 1$ $\quad b^{\log_b x} = x$

SOLVE LOGARITHMIC EQUATIONS AND INEQUALITIES

A **logarithmic equation** is an equation that contains one or more logarithms. You can use the definition of a logarithm to help you solve logarithmic equations.

Example 5 Solve a Logarithmic Equation

Solve $\log_4 n = \frac{5}{2}$.

$\log_4 n = \frac{5}{2}$ — Original equation

$n = 4^{\frac{5}{2}}$ — Definition of logarithm

$n = (2^2)^{\frac{5}{2}}$ — $4 = 2^2$

$n = 2^5$ — Power of a Power

$n = 32$ — Simplify.

A **logarithmic inequality** is an inequality that involves logarithms. In the case of inequalities, the following property is helpful.

Key Concept — Logarithmic to Exponential Inequality

- **Symbols** If $b > 1$, $x > 0$, and $\log_b x > y$, then $x > b^y$.
  If $b > 1$, $x > 0$, and $\log_b x < y$, then $0 < x < b^y$.
- **Examples** $\log_2 x > 3$, $x > 2^3$ $\qquad$ $\log_3 x < 5$, $0 < x < 3^5$

Example 6 Solve a Logarithmic Inequality

Solve $\log_5 x < 2$. Check your solution.

$\log_5 x < 2$ — Original inequality

$0 < x < 5^2$ — Logarithmic to exponential inequality

$0 < x < 25$ — Simplify.

The solution set is $\{x \mid 0 < x < 25\}$.

CHECK Try 5 to see if it satisfies the inequality.

$\log_5 x < 2$ — Original inequality

$\log_5 5 \overset{?}{<} 2$ — Substitute 5 for x.

$1 < 2$ ✓ — $\log_5 5 = 1$ because $5^1 = 5$.

Study Tip

Special Values

If $b > 0$ and $b \neq 1$, then the following statements are true.

- $\log_b b = 1$ because $b^1 = b$.
- $\log_b 1 = 0$ because $b^0 = 1$.

Use the following property to solve logarithmic equations that have logarithms with the same base on each side.

Key Concept *Property of Equality for Logarithmic Functions*

- **Symbols** If b is a positive number other than 1, then $\log_b x = \log_b y$ if and only if $x = y$.
- **Example** If $\log_7 x = \log_7 3$, then $x = 3$.

Example 7 Solve Equations with Logarithms on Each Side

Solve $\log_5 (p^2 - 2) = \log_5 p$. Check your solution.

$\log_5 (p^2 - 2) = \log_5 p$	Original equation
$p^2 - 2 = p$	Property of Equality for Logarithmic Functions
$p^2 - p - 2 = 0$	Subtract p from each side.
$(p - 2)(p + 1) = 0$	Factor.
$p - 2 = 0$ or $p + 1 = 0$	Zero Product Property
$p = 2$ $\quad$ $p = -1$	Solve each equation.

CHECK Substitute each value into the original equation.

$\log_5 (2^2 - 2) \stackrel{?}{=} \log_5 2$	Substitute 2 for p.
$\log_5 2 = \log_5 2$ ✓	Simplify.
$\log_5 [(-1)^2 - 2] \stackrel{?}{=} \log_5 (-1)$	Substitute -1 for p.

Since $\log_5 (-1)$ is undefined, -1 is an *extraneous* solution and must be eliminated. Thus, the solution is 2.

Study Tip

Extraneous Solutions
The domain of a logarithmic function does not include negative values. For this reason, be sure to check for extraneous solutions of logarithmic equations.

Use the following property to solve logarithmic inequalities that have the same base on each side. Exclude values from your solution set that would result in taking the logarithm of a number less than or equal to zero in the original inequality.

Key Concept *Property of Inequality for Logarithmic Functions*

- **Symbols** If $b > 1$, then $\log_b x > \log_b y$ if and only if $x > y$, and $\log_b x < \log_b y$ if and only if $x < y$.
- **Example** If $\log_2 x > \log_2 9$, then $x > 9$.

This property also holds for $\leq$ and $\geq$.

Example 8 Solve Inequalities with Logarithms on Each Side

Solve $\log_{10} (3x - 4) < \log_{10} (x + 6)$. Check your solution.

$\log_{10} (3x - 4) < \log_{10} (x + 6)$	Original inequality
$3x - 4 < x + 6$	Property of Inequality for Logarithmic Functions
$2x < 10$	Addition and Subtraction Properties of Inequalities
$x < 5$	Divide each side by 2.

We must exclude from this solution all values of x such that $3x - 4 \leq 0$ or $x + 6 \leq 0$. Thus, the solution set is $x > \frac{4}{3}$ and $x > -6$ and $x < 5$. This compound inequality simplifies to $\frac{4}{3} < x < 5$.

Study Tip

Look back
To review **compound inequalities**, see Lesson 1-6.

Check for Understanding

Concept Check

1. **OPEN ENDED** Give an example of an exponential equation and its related logarithmic equation.

2. **Describe** the relationship between $y = 3^x$ and $y = \log_3 x$.

3. **FIND THE ERROR** Paul and Scott are solving $\log_3 x = 9$.

Paul	Scott
$\log_3 x = 9$	$\log_3 x = 9$
$3^x = 9$	$x = 3^9$
$3^x = 3^2$	$x = 19{,}683$
$x = 2$	

Who is correct? Explain your reasoning.

Guided Practice

Write each equation in logarithmic form.

4. $5^4 = 625$

5. $7^{-2} = \frac{1}{49}$

Write each equation in exponential form.

6. $\log_3 81 = 4$

7. $\log_{36} 6 = \frac{1}{2}$

Evaluate each expression.

8. $\log_4 256$

9. $\log_2 \frac{1}{8}$

10. $3^{\log_3 21}$

11. $\log_5 5^{-1}$

Solve each equation or inequality. Check your solutions.

12. $\log_9 x = \frac{3}{2}$

13. $\log_{\frac{1}{10}} x = -3$

14. $\log_3 (2x - 1) \leq 2$

15. $\log_5 (3x - 1) = \log_5 2x^2$

16. $\log_2 (3x - 5) > \log_2 (x + 7)$

17. $\log_b 9 = 2$

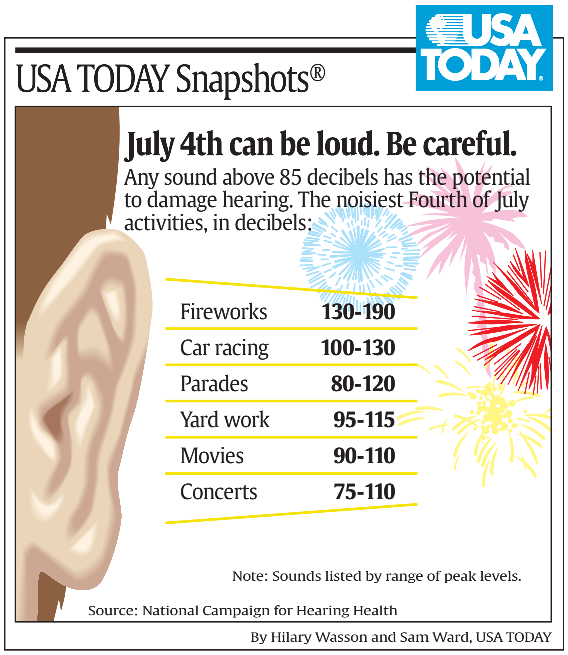
Application

SOUND **For Exercises 18–20, use the following information.**
An equation for loudness L, in decibels, is $L = 10 \log_{10} R$, where R is the relative intensity of the sound.

18. Solve $130 = 10 \log_{10} R$ to find the relative intensity of a fireworks display with a loudness of 130 decibels.

19. Solve $75 = 10 \log_{10} R$ to find the relative intensity of a concert with a loudness of 75 decibels.

20. How many times more intense is the fireworks display than the concert? In other words, find the ratio of their intensities.



Practice and Apply

Homework Help

For Exercises	See Examples
21–26	1
27–32	2
33–46	3
47–62	4–7
63–65	4
68–70	5

Extra Practice

See page 849.

Write each equation in logarithmic form.

21. $8^3 = 512$ **22.** $3^3 = 27$ **23.** $5^{-3} = \frac{1}{125}$

24. $\left(\frac{1}{3}\right)^{-2} = 9$ **25.** $100^{\frac{1}{2}} = 10$ **26.** $2401^{\frac{1}{4}} = 7$

Write each equation in exponential form.

27. $\log_5 125 = 3$ **28.** $\log_{13} 169 = 2$ **29.** $\log_4 \frac{1}{4} = -1$

30. $\log_{100} \frac{1}{10} = -\frac{1}{2}$ **31.** $\log_8 4 = \frac{2}{3}$ **32.** $\log_{\frac{1}{5}} 25 = -2$

Evaluate each expression.

33. $\log_2 16$ **34.** $\log_{12} 144$ **35.** $\log_{16} 4$

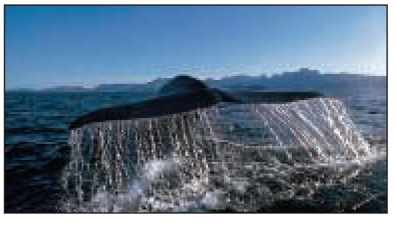
36. $\log_9 243$ **37.** $\log_2 \frac{1}{32}$ **38.** $\log_3 \frac{1}{81}$

39. $\log_5 5^7$ **40.** $2^{\log_2 45}$ **41.** $\log_{11} 11^{(n-5)}$

42. $6^{\log_6 (3x+2)}$ **43.** $\log_{10} 0.001$ **44.** $\log_4 16^x$

WORLD RECORDS **For Exercises 45 and 46, use the information given for Exercises 18–20 to find the relative intensity of each sound.** Source: *The Guinness Book of Records*

45. The loudest animal sounds are the low-frequency pulses made by blue whales when they communicate. These pulses have been measured up to 188 decibels.



46. The loudest insect is the African cicada. It produces a calling song that measures 106.7 decibels at a distance of 50 centimeters.

Solve each equation or inequality. Check your solutions.

47. $\log_9 x = 2$ **48.** $\log_2 c > 8$

49. $\log_{64} y \leq \frac{1}{2}$ **50.** $\log_{25} n = \frac{3}{2}$

51. $\log_{\frac{1}{7}} x = -1$ **52.** $\log_{\frac{1}{3}} p < 0$

53. $\log_2 (3x - 8) \geq 6$ **54.** $\log_{10} (x^2 + 1) = 1$

55. $\log_b 64 = 3$ **56.** $\log_b 121 = 2$

57. $\log_5 5^{6n+1} = 13$ **58.** $\log_5 x = \frac{1}{2}$

59. $\log_6 (2x - 3) = \log_6 (x + 2)$ **60.** $\log_2 (4y - 10) \geq \log_2 (y - 1)$

61. $\log_{10} (a^2 - 6) > \log_{10} a$ **62.** $\log_7 (x^2 + 36) = \log_7 100$

Show that each statement is true.

63. $\log_5 25 = 2 \log_5 5$ **64.** $\log_{16} 2 \cdot \log_2 16 = 1$ **65.** $\log_7 [\log_3 (\log_2 8)] = 0$

66. **a.** Sketch the graphs of $y = \log_{\frac{1}{2}} x$ and $y = \left(\frac{1}{2}\right)^x$ on the same axes.

b. Describe the relationship between the graphs.

67. **a.** Sketch the graphs of $y = \log_2 x + 3$, $y = \log_2 x - 4$, $y = \log_2 (x - 1)$, and $y = \log_2 (x + 2)$.

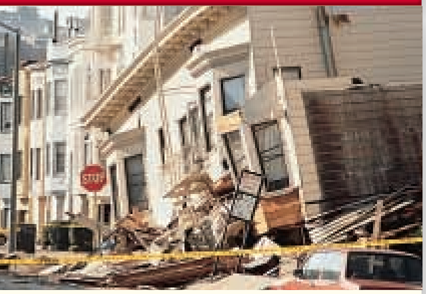
b. Describe this family of graphs in terms of its parent graph $y = \log_2 x$.

EARTHQUAKE **For Exercises 68 and 69, use the following information.**
The magnitude of an earthquake is measured on a logarithmic scale called the Richter scale. The magnitude M is given by $M = \log_{10} x$, where x represents the amplitude of the seismic wave causing ground motion.

68. How many times as great is the amplitude caused by an earthquake with a Richter scale rating of 7 as an aftershock with a Richter scale rating of 4?

69. How many times as great was the motion caused by the 1906 San Francisco earthquake that measured 8.3 on the Richter scale as that caused by the 2001 Bhuj, India, earthquake that measured 6.9?

More About. . .



Earthquake

The Loma Prieta earthquake measured 7.1 on the Richter scale and interrupted the 1989 World Series in San Francisco.

Source: U.S. Geological Survey

70. **NOISE ORDINANCE** A proposed city ordinance will make it illegal to create sound in a residential area that exceeds 72 decibels during the day and 55 decibels during the night. How many times more intense is the noise level allowed during the day than at night?

71. **CRITICAL THINKING** The value of $\log_2 5$ is between two consecutive integers. Name these integers and explain how you determined them.

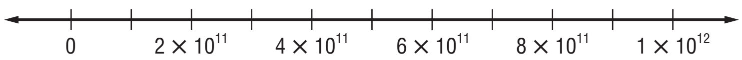
72. **CRITICAL THINKING** Using the definition of a logarithmic function where $y = \log_b x$, explain why the base b cannot equal 1.

73. **WRITING IN MATH** Answer the question that was posed at the beginning of the lesson.

Why is a logarithmic scale used to measure sound?

Include the following in your answer:

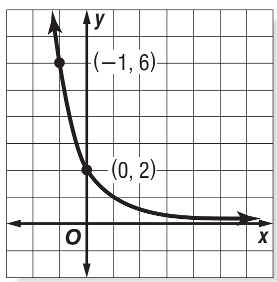
- the relative intensities of a pin drop, a whisper, normal conversation, kitchen noise, and a jet engine written in scientific notation,
- a plot of each of these relative intensities on the scale shown below, and



- an explanation as to why the logarithmic scale might be preferred over the scale shown above.

Standardized Test Practice

74. What is the equation of the function graphed at the right?



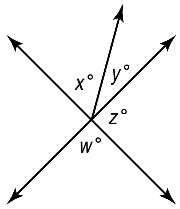
Ⓐ $y = 2(3)^x$

Ⓑ $y = 2\left(\frac{1}{3}\right)^x$

Ⓒ $y = 3\left(\frac{1}{2}\right)^x$

Ⓓ $y = 3(2)^x$

75. In the figure at the right, if $y = \frac{2}{7}x$ and $z = 3w$, then $x =$



Ⓐ 14. Ⓑ 20.

Ⓒ 28. Ⓓ 35.

Maintain Your Skills

Mixed Review

Simplify each expression. *(Lesson 10-1)*

76. $x^{\sqrt{6}} \cdot x^{\sqrt{6}}$

77. $\left(b^{\sqrt{6}}\right)^{\sqrt{24}}$

Solve each equation. Check your solutions. *(Lesson 9-6)*

78. $\frac{2x+1}{x} - \frac{x+1}{x-4} = \frac{-20}{x^2 - 4x}$

79. $\frac{2a-5}{a-9} - \frac{a-3}{3a+2} = \frac{5}{3a^2 - 25a - 18}$

Solve each equation by using the method of your choice. Find exact solutions. *(Lesson 6-5)*

80. $9y^2 = 49$

81. $2p^2 = 5p + 6$

Simplify each expression. *(Lesson 9-2)*

82. $\frac{3}{2y} + \frac{4}{3y} - \frac{7}{5y}$

83. $\frac{x-7}{x^2-9} - \frac{x-3}{x^2+10x+21}$

84. BANKING Donna Bowers has \$4000 she wants to save in the bank. A certificate of deposit (CD) earns 8% annual interest, while a regular savings account earns 3% annual interest. Ms. Bowers doesn't want to tie up all her money in a CD, but she has decided she wants to earn \$240 in interest for the year. How much money should she put in to each type of account? (*Hint:* Use Cramer's Rule.) *(Lesson 4-4)*

Getting Ready for the Next Lesson

PREREQUISITE SKILL **Simplify. Assume that no variable equals zero.**
*(To review **multiplying and dividing monomials**, see Lesson 5-1.)*

85. $x^4 \cdot x^6$

86. $(y^3)^8$

87. $(2a^2b)^3$

88. $\frac{a^4n^7}{a^3n}$

89. $\frac{x^5yz^2}{x^2y^3z^5}$

90. $\left(\frac{b^7}{a^4}\right)^0$

Practice Quiz 1

Lessons 10-1 and 10-2

1. Determine whether $5(1.2)^x$ represents exponential *growth* or *decay*. *(Lesson 10-1)*

2. Write an exponential function whose graph passes through (0, 2) and (2, 32).

3. Write an equivalent logarithmic equation for $4^6 = 4096$. *(Lesson 10-2)*

4. Write an equivalent exponential equation for $\log_9 27 = \frac{3}{2}$. *(Lesson 10-2)*

Evaluate each expression. *(Lesson 10-2)*

5. $\log_8 16$

6. $\log_4 4^{15}$

Solve each equation or inequality. Check your solutions. *(Lessons 10-1 and 10-2)*

7. $3^{4x} = 3^{3-x}$

8. $3^{2n} \leq \frac{1}{9}$

9. $\log_2 (x + 6) > 5$

10. $\log_5 (4x - 1) = \log_5 (3x + 2)$

Graphing Calculator Investigation

A Follow-Up of Lesson 10-2

Modeling Real-World Data: Curve Fitting

We are often confronted with data for which we need to find an equation that best fits the information. We can find exponential and logarithmic functions of best fit using a TI-83 Plus graphing calculator.

Example

The population per square mile in the United States has changed dramatically over a period of years. The table shows the number of people per square mile for several years.

a. Use a graphing calculator to enter the data and draw a scatter plot that shows how the number of people per square mile is related to the year.

U.S. Population Density			
Year	People per square mile	Year	People per square mile
1790	4.5	1900	21.5
1800	6.1	1910	26.0
1810	4.3	1920	29.9
1820	5.5	1930	34.7
1830	7.4	1940	37.2
1840	9.8	1950	42.6
1850	7.9	1960	50.6
1860	10.6	1970	57.5
1870	10.9	1980	64.0
1880	14.2	1990	70.3
1890	17.8	2000	80.0

Source: Northeast-Midwest Institute

Step 1 Enter the year into L1 and the people per square mile into L2.

KEYSTROKES: *See pages 87 and 88 to review how to enter lists.*

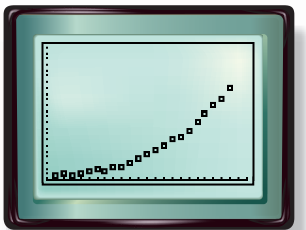
Be sure to clear the Y= list. Use the ▶ key to move the cursor from L1 to L2.

Step 2 Draw the scatter plot.

KEYSTROKES: *See pages 87 and 88 to review how to graph a scatter plot.*

Make sure that **Plot 1** is on, the scatter plot is chosen, **Xlist** is L1, and **Ylist** is L2. Use the viewing window [1780, 2020] with a scale factor of 10 by [0, 115] with a scale factor of 5.

We see from the graph that the equation that best fits the data is a curve. Based on the shape of the curve, try an exponential model.



[1780, 2020] scl: 10 by [0, 115] scl: 5

Step 3 To determine the exponential equation that best fits the data, use the exponential regression feature of the calculator.

KEYSTROKES: STAT ▶ 0 2nd [L1] , 2nd [L2] ENTER

The equation is $y = 1.835122 \times 10^{-11}(1.014700091)^x$.

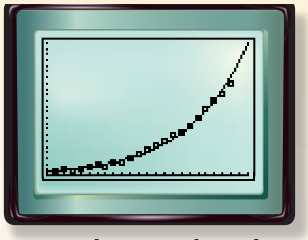
(continued on the next page)

www.algebra2.com/other_calculator_keystrokes

The calculator also reports an r value of 0.991887235. Recall that this number is a correlation coefficient that indicates how well the equation fits the data. A perfect fit would be $r = 1$. Therefore, we can conclude that this equation is a pretty good fit for the data.

To check this equation visually, overlap the graph of the equation with the scatter plot.

KEYSTROKES: Y= VARS 5 ▶ ▶ 1 GRAPH

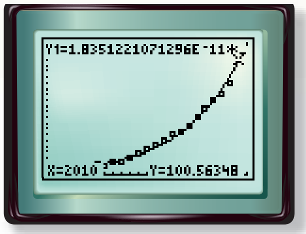


[1780, 2020] scl: 10 by [0, 115] scl: 5

b. If this trend continues, what will be the population per square mile in 2010?

To determine the population per square mile in 2010, from the graphics screen, find the value of y when $x = 2010$.

KEYSTROKES: 2nd [CALC] 1 2010 ENTER



[1780, 2020] scl: 10 by [0, 115] scl: 5

The calculator returns a value of approximately 100.6. If this trend continues, in 2010, there will be approximately 100.6 people per square mile.

Exercises

In 1985, Erika received $30 from her aunt and uncle for her seventh birthday. Her father deposited it into a bank account for her. Both Erika and her father forgot about the money and made no further deposits or withdrawals. The table shows the account balance for several years.

Elapsed Time (years)	Balance
0	$30.00
5	$41.10
10	$56.31
15	$77.16
20	$105.71
25	$144.83
30	$198.43

1. Use a graphing calculator to draw a scatter plot for the data.
2. Calculate and graph the curve of best fit that shows how the elapsed time is related to the balance. Use ExpReg for this exercise.
3. Write the equation of best fit.
4. Write a sentence that describes the fit of the graph to the data.
5. Based on the graph, estimate the balance in 41 years. Check this using the CALC value.
6. Do you think there are any other types of equations that would be good models for these data? Why or why not?

10-3 Properties of Logarithms

What You'll Learn

- Simplify and evaluate expressions using the properties of logarithms.
- Solve logarithmic equations using the properties of logarithms.

How are the properties of exponents and logarithms related?

In Lesson 5-1, you learned that the product of powers is the sum of their exponents.

$$9 \cdot 81 = 3^2 \cdot 3^4 \text{ or } 3^{2+4}$$

In Lesson 10-2, you learned that logarithms *are* exponents, so you might expect that a similar property applies to logarithms. Let's consider a specific case. Does $\log_3 (9 \cdot 81) = \log_3 9 + \log_3 81$?

$\log_3 (9 \cdot 81) = \log_3 (3^2 \cdot 3^4)$ Replace 9 with 3^2 and 81 with 3^4.

$= \log_3 3^{(2+4)}$ Product of Powers

$= 2 + 4$ or 6 Inverse property of exponents and logarithms

$\log_3 9 + \log_3 81 = \log_3 3^2 + \log_3 3^4$ Replace 9 with 3^2 and 81 with 3^4.

$= 2 + 4$ or 6 Inverse property of exponents and logarithms

So, $\log_3 (9 \cdot 81) = \log_3 9 + \log_3 81$.

PROPERTIES OF LOGARITHMS Since logarithms are exponents, the properties of logarithms can be derived from the properties of exponents. The example above and other similar examples suggest the following property of logarithms.

Key Concept — *Product Property of Logarithms*

- **Words** The logarithm of a product is the sum of the logarithms of its factors.
- **Symbols** For all positive numbers m, n, and b, where $b \neq 1$, $\log_b mn = \log_b m + \log_b n$.
- **Example** $\log_3 (4)(7) = \log_3 4 + \log_3 7$

To show that this property is true, let $b^x = m$ and $b^y = n$. Then, using the definition of logarithm, $x = \log_b m$ and $y = \log_b n$.

$b^x b^y = mn$

$b^{x+y} = mn$ Product of Powers

$\log_b b^{x+y} = \log_b mn$ Property of Equality for Logarithmic Functions

$x + y = \log_b mn$ Inverse Property of Exponents and Logarithms

$\log_b m + \log_b n = \log_b mn$ Replace x with $\log_b m$ and y with $\log_b n$.

You can use the Product Property of Logarithms to approximate logarithmic expressions.

Example 1 Use the Product Property

Use $\log_2 3 \approx 1.5850$ to approximate the value of $\log_2 48$.

$\log_2 48 = \log_2 (2^4 \cdot 3)$ — Replace 48 with $16 \cdot 3$ or $2^4 \cdot 3$.

$= \log_2 2^4 + \log_2 3$ — Product Property

$= 4 + \log_2 3$ — Inverse Property of Exponents and Logarithms

$\approx 4 + 1.5850$ or 5.5850 — Replace $\log_2 3$ with 1.5850.

Thus, $\log_2 48$ is approximately 5.5850.

Recall that the quotient of powers is found by subtracting exponents. The property for the logarithm of a quotient is similar.

Key Concept — Quotient Property of Logarithms

- **Words** The logarithm of a quotient is the difference of the logarithms of the numerator and the denominator.
- **Symbols** For all positive numbers m, n, and b, where $b \neq 1$, $\log_b \frac{m}{n} = \log_b m - \log_b n$.

You will show that this property is true in Exercise 47.

Career Choices

Sound Technician

Sound technicians produce movie sound tracks in motion picture production studios, control the sound of live events such as concerts, or record music in a recording studio.

Online Research
For information about a career as a sound technician, visit: www.algebra2.com/careers

Example 2 Use the Quotient Property

Use $\log_3 5 \approx 1.4650$ and $\log_3 20 \approx 2.7268$ to approximate $\log_3 4$.

$\log_3 4 = \log_3 \frac{20}{5}$ — Replace 4 with the quotient $\frac{20}{5}$.

$= \log_3 20 - \log_3 5$ — Quotient Property

$\approx 2.7268 - 1.4650$ or 1.2618 — $\log_3 20 = 2.7268$ and $\log_3 5 = 1.4650$

Thus, $\log_3 4$ is approximately 1.2618.

CHECK Using the definition of logarithm and a calculator, $3^{1.2618} \approx 4$. ✓

Example 3 Use Properties of Logarithms

SOUND **The loudness L of a sound in decibels is given by $L = 10 \log_{10} R$, where R is the sound's relative intensity. Suppose one person talks with a relative intensity of 10^6 or 60 decibels. Would the sound of ten people each talking at that same intensity be ten times as loud or 600 decibels? Explain your reasoning.**

Let L_1 be the loudness of one person talking. $\rightarrow$ $L_1 = 10 \log_{10} 10^6$
Let L_2 be the loudness of ten people talking. $\rightarrow$ $L_2 = 10 \log_{10} (10 \cdot 10^6)$

Then the increase in loudness is $L_2 - L_1$.

$L_2 - L_1 = 10 \log_{10} (10 \cdot 10^6) - 10 \log_{10} 10^6$ — Substitute for L_1 and L_2.

$= 10(\log_{10} 10 + \log_{10} 10^6) - 10 \log_{10} 10^6$ — Product Property

$= 10 \log_{10} 10 + 10 \log_{10} 10^6 - 10 \log_{10} 10^6$ — Distributive Property

$= 10 \log_{10} 10$ — Subtract.

$= 10(1)$ or 10 — Inverse Property of Exponents and Logarithms

The sound of two people talking is perceived by the human ear to be only about 10 decibels louder than the sound of one person talking, or 70 decibels.

Recall that the power of a power is found by multiplying exponents. The property for the logarithm of a power is similar.

Key Concept — *Power Property of Logarithms*

- **Words** The logarithm of a power is the product of the logarithm and the exponent.
- **Symbols** For any real number p and positive numbers m and b, where $b \neq 1$, $\log_b m^p = p \log_b m$.

You will show that this property is true in Exercise 50.

Example 4 Power Property of Logarithms

Given $\log_4 6 \approx 1.2925$, approximate the value of $\log_4 36$.

$\log_4 36 = \log_4 6^2$ Replace 36 with 6^2.

$= 2 \log_4 6$ Power Property

$\approx 2(1.2925)$ or 2.585 Replace $\log_4 6$ with 1.2925.

SOLVE LOGARITHMIC EQUATIONS You can use the properties of logarithms to solve equations involving logarithms.

Example 5 Solve Equations Using Properties of Logarithms

Solve each equation.

a. $3 \log_5 x - \log_5 4 = \log_5 16$

$3 \log_5 x - \log_5 4 = \log_5 16$ Original equation

$\log_5 x^3 - \log_5 4 = \log_5 16$ Power Property

$\log_5 \frac{x^3}{4} = \log_5 16$ Quotient Property

$\frac{x^3}{4} = 16$ Property of Equality for Logarithmic Functions

$x^3 = 64$ Multiply each side by 4.

$x = 4$ Take the cube root of each side.

The solution is 4.

b. $\log_4 x + \log_4 (x - 6) = 2$

$\log_4 x + \log_4 (x - 6) = 2$ Original equation

$\log_4 x(x - 6) = 2$ Product Property

$x(x - 6) = 4^2$ Definition of logarithm

$x^2 - 6x - 16 = 0$ Subtract 16 from each side.

$(x - 8)(x + 2) = 0$ Factor.

$x - 8 = 0$ or $x + 2 = 0$ Zero Product Property

$x = 8$ $\quad$ $x = -2$ Solve each equation.

CHECK Substitute each value into the original equation.

$\log_4 8 + \log_4 (8 - 6) \stackrel{?}{=} 2$

$\log_4 8 + \log_4 2 \stackrel{?}{=} 2$

$\log_4 (8 \cdot 2) \stackrel{?}{=} 2$

$\log_4 16 \stackrel{?}{=} 2$

$2 = 2$ ✓

$\log_4 (-2) + \log_4 (-2 - 6) \stackrel{?}{=} 2$

$\log_4 (-2) + \log_4 (-8) \stackrel{?}{=} 2$

Since $\log_4 (-2)$ and $\log_4 (-8)$ are undefined, -2 is an extraneous solution and must be eliminated.

The only solution is 8.

Study Tip

Checking Solutions

It is wise to check all solutions to see if they are valid since the domain of a logarithmic function is not the complete set of real numbers.

Check for Understanding

Concept Check

1. **Name** the properties that are used to derive the properties of logarithms.
2. **OPEN ENDED** Write an expression that can be simplified by using two or more properties of logarithms. Then simplify it.
3. **FIND THE ERROR** Umeko and Clemente are simplifying $\log_7 6 + \log_7 3 - \log_7 2$.

Umeko

$\log_7 6 + \log_7 3 - \log_7 2$

$= \log_7 18 - \log_7 2$

$= \log_7 9$

Clemente

$\log_7 6 + \log_7 3 - \log_7 2$

$= \log_7 9 - \log_7 2$

$= \log_7 7$ or 1

Who is correct? Explain your reasoning.

Guided Practice

Use $\log_3 2 \approx 0.6310$ and $\log_3 7 \approx 1.7712$ to approximate the value of each expression.

4. $\log_3 \frac{7}{2}$
5. $\log_3 18$
6. $\log_3 \frac{2}{3}$

Solve each equation. Check your solutions.

7. $\log_3 42 - \log_3 n = \log_3 7$
8. $\log_2 3x + \log_2 5 = \log_2 30$
9. $2 \log_5 x = \log_5 9$
10. $\log_{10} a + \log_{10} (a + 21) = 2$

Application

MEDICINE **For Exercises 11 and 12, use the following information.**
The pH of a person's blood is given by pH $= 6.1 + \log_{10} B - \log_{10} C$, where B is the concentration of bicarbonate, which is a base, in the blood and C is the concentration of carbonic acid in the blood.

11. Use the Quotient Property of Logarithms to simplify the formula for blood pH.
12. Most people have a blood pH of 7.4. What is the approximate ratio of bicarbonate to carbonic acid for blood with this pH?

Practice and Apply

Homework Help

For Exercises	See Examples
13–20	1, 2, 4
21–34	5
37–45	3

Extra Practice

See page 850.

Use $\log_5 2 \approx 0.4307$ and $\log_5 3 \approx 0.6826$ to approximate the value of each expression.

13. $\log_5 9$
14. $\log_5 8$
15. $\log_5 \frac{2}{3}$
16. $\log_5 \frac{3}{2}$
17. $\log_5 50$
18. $\log_5 30$
19. $\log_5 0.5$
20. $\log_5 \frac{10}{9}$

Solve each equation. Check your solutions.

21. $\log_3 5 + \log_3 x = \log_3 10$
22. $\log_4 a + \log_4 9 = \log_4 27$
23. $\log_{10} 16 - \log_{10} 2t = \log_{10} 2$
24. $\log_7 24 - \log_7 (y + 5) = \log_7 8$
25. $\log_2 n = \frac{1}{4} \log_2 16 + \frac{1}{2} \log_2 49$
26. $2 \log_{10} 6 - \frac{1}{3} \log_{10} 27 = \log_{10} x$
27. $\log_{10} z + \log_{10} (z + 3) = 1$
28. $\log_6 (a^2 + 2) + \log_6 2 = 2$
29. $\log_2 (12b - 21) - \log_2 (b^2 - 3) = 2$
30. $\log_2 (y + 2) - \log_2 (y - 2) = 1$
31. $\log_3 0.1 + 2 \log_3 x = \log_3 2 + \log_3 5$
32. $\log_5 64 - \log_5 \frac{8}{3} + \log_5 2 = \log_5 4p$

CONTENTS

Solve for *n*.

33. $\log_a 4n - 2\log_a x = \log_a x$

34. $\log_b 8 + 3\log_b n = 3\log_b (x - 1)$

CRITICAL THINKING **Tell whether each statement is *true* or *false*. If true, show that it is true. If false, give a counterexample.**

35. For all positive numbers m, n, and b, where $b \neq 1$, $\log_b (m + n) = \log_b m + \log_b n$.

36. For all positive numbers m, n, x, and b, where $b \neq 1$, $n \log_b x + m \log_b x = (n + m) \log_b x$.

37. **EARTHQUAKES** The great Alaskan earthquake in 1964 was about 100 times more intense than the Loma Prieta earthquake in San Francisco in 1989. Find the difference in the Richter scale magnitudes of the earthquakes.

BIOLOGY **For Exercises 38–40, use the following information.**
The energy E (in kilocalories per gram molecule) needed to transport a substance from the outside to the inside of a living cell is given by $E = 1.4(\log_{10} C_2 - \log_{10} C_1)$, where C_1 is the concentration of the substance outside the cell and C_2 is the concentration inside the cell.

38. Express the value of E as one logarithm.

39. Suppose the concentration of a substance inside the cell is twice the concentration outside the cell. How much energy is needed to transport the substance on the outside of the cell to the inside? (Use $\log_{10} 2 \approx 0.3010$.)

40. Suppose the concentration of a substance inside the cell is four times the concentration outside the cell. How much energy is needed to transport the substance from the outside of the cell to the inside?

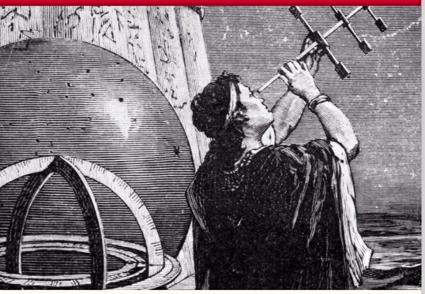
SOUND **For Exercises 41–43, use the formula for the loudness of sound in Example 3 on page 542. Use $\log_{10} 2 \approx 0.3010$ and $\log_{10} 3 \approx 0.47712$.**

41. A certain sound has a relative intensity of R. By how many decibels does the sound increase when the intensity is doubled?

42. A certain sound has a relative intensity of R. By how many decibels does the sound decrease when the intensity is halved?

43. A stadium containing 10,000 cheering people can produce a crowd noise of about 90 decibels. If every one cheers with the same relative intensity, how much noise, in decibels, is a crowd of 30,000 people capable of producing? Explain your reasoning.

More About. . .



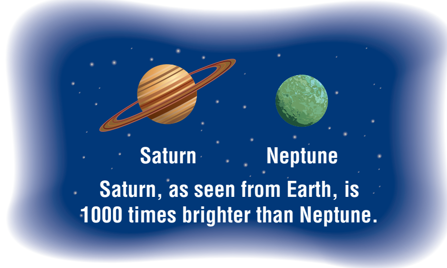
Star Light

The Greek astronomer Hipparchus made the first known catalog of stars. He listed the brightness of each star on a scale of 1 to 6, the brightest being 1. With no telescope, he could only see stars as dim as the 6th magnitude.

Source: NASA

STAR LIGHT **For Exercises 44–46, use the following information.**
The brightness, or apparent magnitude, m of a star or planet is given by the formula $m = 6 - 2.5 \log_{10} \frac{L}{L_0}$, where L is the amount of light coming to Earth from the star or planet and L_0 is the amount of light from a sixth magnitude star.

The crescent moon is about 100 times brighter than the brightest star, Sirius.



Saturn, as seen from Earth, is 1000 times brighter than Neptune.

44. Find the difference in the magnitudes of Sirius and the crescent moon.

45. Find the difference in the magnitudes of Saturn and Neptune.

46. **RESEARCH** Use the Internet or other reference to find the magnitude of the dimmest stars that we can now see with ground-based telescopes.

47. **CRITICAL THINKING** Use the properties of exponents to prove the Quotient Property of Logarithms.

48. **WRITING IN MATH** Answer the question that was posed at the beginning of the lesson.

How are the properties of exponents and logarithms related?

Include the following in your answer:

- examples like the one shown at the beginning of the lesson illustrating the Quotient Property and Power Property of Logarithms, and
- an explanation of the similarity between one property of exponents and its related property of logarithms.

Standardized Test Practice

49. Simplify $2 \log_5 12 - \log_5 8 - 2 \log_5 3$.

(A) $\log_5 2$ (B) $\log_5 3$ (C) $\log_5 0.5$ (D) 1

50. **SHORT RESPONSE** Show that $\log_b m^p = p \log_b m$ for any real number p and positive number m and b, where $b \neq 1$.

Maintain Your Skills

Mixed Review

Evaluate each expression. *(Lesson 10-2)*

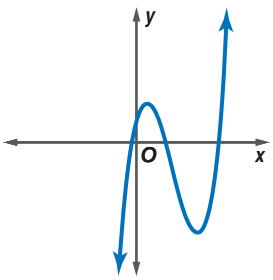
51. $\log_3 81$ 52. $\log_9 \frac{1}{729}$ 53. $\log_7 7^{2x}$

Solve each equation or inequality. Check your solutions. *(Lesson 10-1)*

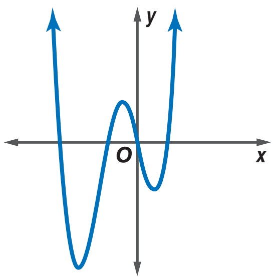
54. $3^{5n+3} = 3^{33}$ 55. $7^a = 49^{-4}$ 56. $3^{d+4} > 9^d$

Determine whether each graph represents an odd-degree polynomial function or an even-degree polynomial function. Then state how many real zeros each function has. *(Lesson 7-1)*

57.



58.



Simplify each expression. *(Lesson 9-1)*

59. $\frac{39a^3b^4}{13a^4b^3}$ 60. $\frac{k+3}{5kl} \cdot \frac{10kl}{k+3}$ 61. $\frac{5y - 15z}{42x^2} \div \frac{y - 3z}{14x}$

62. **PHYSICS** If a stone is dropped from a cliff, the equation $t = \frac{1}{4}\sqrt{d}$ represents the time t in seconds that it takes for the stone to reach the ground. If d represents the distance in feet that the stone falls, find how long it would take for a stone to fall from a 150-foot cliff. *(Lesson 5-6)*

Getting Ready for the Next Lesson

PREREQUISITE SKILL **Solve each equation or inequality. Check your solutions.**

*(To review **solving logarithmic equations and inequalities**, see Lesson 10-2.)*

63. $\log_3 x = \log_3 (2x - 1)$ 64. $\log_{10} 2^x = \log_{10} 32$

65. $\log_2 3x > \log_2 5$ 66. $\log_5 (4x + 3) < \log_5 11$

10-4 Common Logarithms

What You'll Learn

- Solve exponential equations and inequalities using common logarithms.
- Evaluate logarithmic expressions using the Change of Base Formula.

Vocabulary

- common logarithm
- Change of Base Formula

Why is a logarithmic scale used to measure acidity?

The pH level of a substance measures its acidity. A low pH indicates an acid solution while a high pH indicates a basic solution. The pH levels of some common substances are shown.

The pH level of a substance is given by pH $= -\log_{10} [H+]$, where $H+$ is the substance's hydrogen ion concentration in moles per liter. Another way of writing this formula is pH $= -\log [H+]$.

Acidity of Common Substances

Substance	pH Level
Battery acid	1.0
Sauerkraut	3.5
Tomatoes	4.2
Black Coffee	5.0
Milk	6.4
Distilled Water	7.0
Eggs	7.8
Milk of magnesia	10.0

COMMON LOGARITHMS You have seen that the base 10 logarithm function, $y = \log_{10} x$, is used in many applications. Base 10 logarithms are called **common logarithms**. Common logarithms are usually written without the subscript 10.

$$\log_{10} x = \log x, x > 0$$

Most calculators have a LOG key for evaluating common logarithms.

Study Tip

Technology
Nongraphing scientific calculators often require entering the number followed by the function, for example, 3 LOG.

Example 1 Find Common Logarithms

Use a calculator to evaluate each expression to four decimal places.

a. $\log 3$ KEYSTROKES: LOG 3 ENTER .4771212547 about 0.4771

b. $\log 0.2$ KEYSTROKES: LOG 0.2 ENTER −.6989700043 about −0.6990

Sometimes an application of logarithms requires that you use the inverse of logarithms, or exponentiation.

$$10^{\log x} = x$$

Example 2 Solve Logarithmic Equations Using Exponentiation

EARTHQUAKES The amount of energy E, in ergs, that an earthquake releases is related to its Richter scale magnitude M by the equation $\log E = 11.8 + 1.5M$. The Chilean earthquake of 1960 measured 8.5 on the Richter scale. How much energy was released?

$\log E = 11.8 + 1.5M$ Write the formula.

$\log E = 11.8 + 1.5(8.5)$ Replace M with 8.5.

$\log E = 24.55$ Simplify.

$10^{\log E} = 10^{24.55}$ Write each side using exponents and base 10.

$E = 10^{24.55}$ Inverse Property of Exponents and Logarithms

$E \approx 3.55 \times 10^{24}$ Use a calculator.

The amount of energy released by this earthquake was about 3.55×10^{24} ergs.

Example 3 Solve Exponential Equations Using Logarithms

Solve $3^x = 11$.

$3^x = 11$	Original equation
$\log 3^x = \log 11$	Property of Equality for Logarithmic Functions
$x \log 3 = \log 11$	Power Property of Logarithms
$x = \frac{\log 11}{\log 3}$	Divide each side by log 3.
$x \approx \frac{1.0414}{0.4771}$	Use a calculator.
$x \approx 2.1828$	The solution is approximately 2.1828.

CHECK You can check this answer using a calculator or by using estimation. Since $3^2 = 9$ and $3^3 = 27$, the value of x is between 2 and 3. In addition, the value of x should be closer to 2 than 3, since 11 is closer to 9 than 27. Thus, 2.1828 is a reasonable solution. ✓

Study Tip

Using Logarithms

When you use the Property for Logarithmic Functions as in the second step of Example 3, this is sometimes referred to as *taking the logarithm of each side.*

Example 4 Solve Exponential Inequalities Using Logarithms

Solve $5^{3y} < 8^{y-1}$.

$5^{3y} < 8^{y-1}$	Original inequality
$\log 5^{3y} < \log 8^{y-1}$	Property of Inequality for Logarithmic Functions
$3y \log 5 < (y-1) \log 8$	Power Property of Logarithms
$3y \log 5 < y \log 8 - \log 8$	Distributive Property
$3y \log 5 - y \log 8 < -\log 8$	Subtract $y \log 8$ from each side.
$y(3 \log 5 - \log 8) < -\log 8$	Distributive Property
$y < \frac{-\log 8}{3 \log 5 - \log 8}$	Divide each side by $3 \log 5 - \log 8$.
$y < \frac{-(0.9031)}{3(0.6990) - 0.9031}$	Use a calculator.
$y < -0.7564$	The solution set is $\{y \mid y < -0.7564\}$.

CHECK Test $y = -1$.

$5^{3y} < 8^{y-1}$	Original inequality
$5^{3(-1)} < 8^{(-1)-1}$	Replace y with 1.
$5^{-3} < 8^{-2}$	Simplify.
$\frac{1}{125} < \frac{1}{64}$ ✓	Negative Exponent Property

CHANGE OF BASE FORMULA The **Change of Base Formula** allows you to write equivalent logarithmic expressions that have different bases.

Key Concept — Change of Base Formula

- **Symbols** For all positive numbers, a, b and n, where $a \neq 1$ and $b \neq 1$,

$$\log_a n = \frac{\log_b n}{\log_b a} \begin{matrix} \leftarrow \text{log base } b \text{ of original number} \\ \leftarrow \text{log base } b \text{ of old base} \end{matrix}$$

- **Example** $\log_5 12 = \frac{\log_{10} 12}{\log_{10} 5}$

To prove this formula, let $\log_a n = x$.

$$a^x = n \quad \text{Definition of logarithm}$$
$$\log_b a^x = \log_b n \quad \text{Property of Equality for Logarithms}$$
$$x \log_b a = \log_b n \quad \text{Power Property of Logarithms}$$
$$x = \frac{\log_b n}{\log_b a} \quad \text{Divide each side by } \log_b a.$$
$$\log_a n = \frac{\log_b n}{\log_b a} \quad \text{Replace } x \text{ with } \log_a n.$$

This formula makes it possible to evaluate a logarithmic expression of any base by translating the expression into one that involves common logarithms.

Example 5 Change of Base Formula

Express $\log_4 25$ in terms of common logarithms. Then approximate its value to four decimal places.

$$\log_4 25 = \frac{\log_{10} 25}{\log_{10} 4} \quad \text{Change of Base Formula}$$
$$\approx 2.3219 \quad \text{Use a calculator.}$$

The value of $\log_4 25$ is approximately 2.3219.

Check for Understanding

Concept Check

1. **Name** the base used by the calculator LOG key. What are these logarithms called?
2. **OPEN ENDED** Give an example of an exponential equation requiring the use of logarithms to solve. Then solve your equation.
3. **Explain** why you must use the Change of Base Formula to find the value of $\log_2 7$ on a calculator.

Guided Practice

Use a calculator to evaluate each expression to four decimal places.

4. log 4
5. log 23
6. log 0.5

Solve each equation or inequality. Round to four decimal places.

7. $9^x = 45$
8. $4^{5n} > 30$
9. $3.1^{a-3} = 9.42$
10. $11^{x^2} = 25.4$
11. $7^{t-2} = 5^t$
12. $4^{p-1} \leq 3^p$

Express each logarithm in terms of common logarithms. Then approximate its value to four decimal places.

13. $\log_7 5$
14. $\log_3 42$
15. $\log_2 9$

Application

16. **DIET** Sandra's doctor has told her to avoid foods with a pH that is less than 4.5. What is the hydrogen ion concentration of foods Sandra is allowed to eat? Use the information at the beginning of the lesson.

Practice and Apply

Use a calculator to evaluate each expression to four decimal places.

17. log 5
18. log 12
19. log 7.2
20. log 2.3
21. log 0.8
22. log 0.03

Homework Help

For Exercises	See Examples
17–22	1
23–44, 53–57	3, 4
45–50	5
51–55	2

Extra Practice

See page 850.

ACIDITY **For Exercises 23–26, use the information at the beginning of the lesson to find the pH of each substance given its concentration of hydrogen ions.**

23. ammonia: $[H+] = 1 \times 10^{-11}$ mole per liter

24. vinegar: $[H+] = 6.3 \times 10^{-3}$ mole per liter

25. lemon juice: $[H+] = 7.9 \times 10^{-3}$ mole per liter

26. orange juice: $[H+] = 3.16 \times 10^{-4}$ mole per liter

Solve each equation or inequality. Round to four decimal places.

27. $6^x \geq 42$

28. $5^x = 52$

29. $8^{2a} < 124$

30. $4^{3p} = 10$

31. $3^{n+2} = 14.5$

32. $9^{z-4} = 6.28$

33. $8.2^{n-3} = 42.5$

34. $2.1^{t-5} = 9.32$

35. $20^{x^2} = 70$

36. $2^{x^2-3} = 15$

37. $8^{2n} > 52^{4n+3}$

38. $2^{2x+3} = 3^{3x}$

39. $16^{d-4} = 3^{3-d}$

40. $7^{p+2} \leq 13^{5-p}$

41. $5^{5y-2} = 2^{2y+1}$

42. $8^{2x-5} = 5^{x+1}$

43. $2^n = \sqrt{3^{n-2}}$

44. $4^x = \sqrt{5^{x+2}}$

Express each logarithm in terms of common logarithms. Then approximate its value to four decimal places.

45. $\log_2 13$

46. $\log_5 20$

47. $\log_7 3$

48. $\log_3 8$

49. $\log_4 (1.6)^2$

50. $\log_6 \sqrt{5}$

More About. . .

Pollution

As little as 0.9 milligram per liter of iron at a pH of 5.5 can cause fish to die.

Source: Kentucky Water Watch

For Exercises 51 and 52, use the information presented at the beginning of the lesson.

51. **POLLUTION** The acidity of water determines the toxic effects of runoff into streams from industrial or agricultural areas. A pH range of 6.0 to 9.0 appears to provide protection for freshwater fish. What is this range in terms of the water's hydrogen ion concentration?

52. **BUILDING DESIGN** The 1971 Sylmar earthquake in Los Angeles had a Richter scale magnitude of 6.3. Suppose an architect has designed a building strong enough to withstand an earthquake 50 times as intense as the Sylmar quake. Find the magnitude of the strongest quake this building is designed to withstand.

ASTRONOMY **For Exercises 53–55, use the following information.**
Some stars appear bright only because they are very close to us. Absolute magnitude M is a measure of how bright a star would appear if it were 10 parsecs, about 32 light years, away from Earth. A lower magnitude indicates a brighter star. Absolute magnitude is given by $M = m + 5 - 5 \log d$, where d is the star's distance from Earth measured in parsecs and m is its apparent magnitude.

53. Sirius and Vega are two of the brightest stars in Earth's sky. The apparent magnitude of Sirius is -1.44 and of Vega is 0.03. Which star appears brighter?

54. Sirius is 2.64 parsecs from Earth while Vega is 7.76 parsecs from Earth. Find the absolute magnitude of each star.

55. Which star is actually brighter? That is, which has a lower absolute magnitude?

56. **CRITICAL THINKING**

a. Without using a calculator, find the value of $\log_2 8$ and $\log_8 2$.

b. Without using a calculator, find the value of $\log_9 27$ and $\log_{27} 9$.

c. Make and prove a conjecture as to the relationship between $\log_a b$ and $\log_b a$.

CONTENTS

MONEY **For Exercises 57 and 58, use the following information.**
If you deposit P dollars into a bank account paying an annual interest rate r (expressed as a decimal), with n interest payments each year, the amount A you would have after t years is $A = P\left(1 + \frac{r}{n}\right)^{nt}$. Marta places \$100 in a savings account earning 6% annual interest, compounded quarterly.

57. If Marta adds no more money to the account, how long will it take the money in the account to reach \$125?

58. How long will it take for Marta's money to double?

59. WRITING IN MATH Answer the question that was posed at the beginning of the lesson.

Why is a logarithmic scale used to measure acidity?

Include the following in your answer:

- the hydrogen ion concentration of three substances listed in the table, and
- an explanation as to why it is important to be able to distinguish between a hydrogen ion concentration of 0.00001 mole per liter and 0.0001 mole per liter.

Standardized Test Practice

A B C D

60. **QUANTITATIVE COMPARISION** **Compare the quantity in Column A and the quantity in Column B. Then determine whether:**

(A) the quantity in Column A is greater,
(B) the quantity in Column B is greater,
(C) the two quantities are equal, or
(D) the relationship cannot be determined from the information given.

Column A	Column B
$\log 10^3$	$\log 10^2$

61. If $2^4 = 3^x$, then what is the value of x?

(A) 0.63 (B) 2.34 (C) 2.52 (D) 4

Maintain Your Skills

Mixed Review

Use $\log_7 2 \approx 0.3562$ and $\log_7 3 \approx 0.5646$ to approximate the value of each expression. *(Lesson 10-3)*

62. $\log_7 16$ **63.** $\log_7 27$ **64.** $\log_7 36$

Solve each equation or inequality. Check your solutions. *(Lesson 10-2)*

65. $\log_4 r = 3$ **66.** $\log_8 z \leq -2$ **67.** $\log_3 (4x - 5) = 5$

68. Use synthetic substitution to find $f(-2)$ for $f(x) = x^3 + 6x - 2$. *(Lesson 7-4)*

Factor completely. If the polynomial is not factorable, write prime. *(Lesson 5-4)*

69. $3d^2 + 2d - 8$ **70.** $42pq - 35p + 18q - 15$ **71.** $13xyz + 3x^2z + 4k$

Getting Ready for the Next Lesson

PREREQUISITE SKILLS **Write an equivalent exponential equation.**
*(For review of **logarithmic equations**, see Lesson 10-2.)*

72. $\log_2 3 = x$ **73.** $\log_3 x = 2$ **74.** $\log_5 125 = 3$

Write an equivalent logarithmic equation.
*(For review of **logarithmic equations**, see Lesson 10-2.)*

75. $5^x = 45$ **76.** $7^3 = x$ **77.** $b^y = x$

Solving Exponential and Logarithmic Equations and Inequalities

You can use a TI-83 Plus graphing calculator to solve exponential and logarithmic equations and inequalities. This can be done by graphing each side of the equation separately and using the **intersect** feature on the calculator.

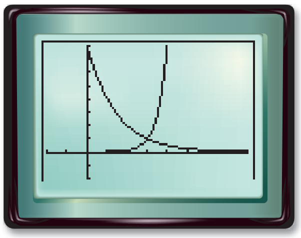
Example 1

Solve $2^{3x-9} = \left(\frac{1}{2}\right)^{x-3}$ by graphing.

Step 1 *Graph each side of the equation.*

- Graph each side of the equation as a separate function. Enter 2^{3x-9} as Y1. Enter $\left(\frac{1}{2}\right)^{x-3}$ as Y2. Be sure to include the added parentheses around each exponent. Then graph the two equations.

KEYSTROKES: *See pages 87 and 88 to review graphing equations.*

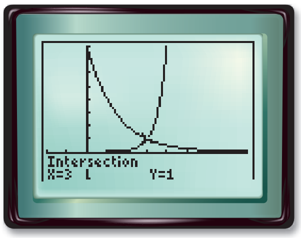


$[-2, 8]$ scl: 1 by $[-2, 8]$ scl: 1

Step 2 *Use the intersect feature.*

- You can use the **intersect** feature on the **CALC** menu to approximate the ordered pair of the point at which the curves cross.

KEYSTROKES: *See page 115 to review how to use the intersect feature.*



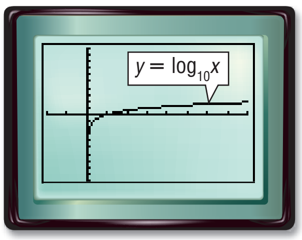
$[-2, 8]$ scl: 1 by $[-2, 8]$ scl: 1

The calculator screen shows that the x-coordinate of the point at which the curves cross is 3. Therefore, the solution of the equation is 3.

The TI-83 Plus has $y = \log_{10} x$ as a built-in function. Enter [Y=] [LOG] [X,T,θ,n] [GRAPH] to view this graph. To graph logarithmic functions with bases other than 10, you must use the Change of Base Formula,

$$\log_a n = \frac{\log_b n}{\log_b a}.$$

For example, $\log_3 x = \frac{\log_{10} x}{\log_{10} 3}$, so to graph $y = \log_3 x$ you must enter [LOG] [X,T,θ,n] [ )] [÷] [LOG] 3 [ )] as Y1.



$[-2, 8]$ scl: 1 by $[-5, 5]$ scl: 1

www.algebra2.com/other_calculator_keystrokes

Investigation

Example 2

Solve $\log_2 2x \geq \log_{\frac{1}{2}} 2x$ by graphing.

Step 1 *Rewrite the problem as a system of common logarithmic inequalities.*

- The first inequality is $\log_2 2x \geq y$ or $y \leq \log_2 2x$. The second inequality is $y \geq \log_{\frac{1}{2}} 2x$.
- Use the Change of Base Formula to create equations that can be entered into the calculator.

$$\log_2 2x = \frac{\log 2x}{\log 2} \qquad \log_{\frac{1}{2}} 2x = \frac{\log 2x}{\log \frac{1}{2}}$$

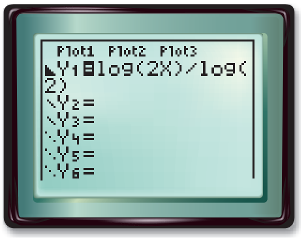
Thus, the two inequalities are $y \leq \frac{\log 2x}{\log 2}$ and $y \geq \frac{\log 2x}{\log \frac{1}{2}}$.

Step 2 *Enter the first inequality.*

- Enter $y \leq \frac{\log 2x}{\log 2}$ as Y1. Since the inequality includes *less than*, shade below the curve.

KEYSTROKES: [Y=] [LOG] 2 [X,T,θ,n] [)] [÷] [LOG] 2 [)]

Use the arrow and [ENTER] keys to choose the shade below icon.

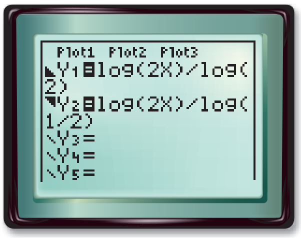


Step 3 *Enter the second inequality.*

- Enter $y \geq \frac{\log 2x}{\log \frac{1}{2}}$ as Y2. Since the inequality includes *greater than*, shade above the curve.

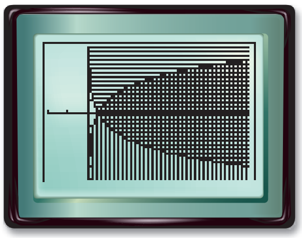
KEYSTROKES: [LOG] 2 [X,T,θ,n] [)] [÷] [LOG] 1 [÷] 2 [)] [GRAPH]

Use the arrow and [ENTER] keys to choose the shade above icon.



Step 4 *Graph the inequalities.*

KEYSTROKES: [GRAPH]



[−2, 8] scl: 1 by [−5, 5] scl: 1

The x values of the points in the region where the shadings overlap is the solution set of the original inequality. Using the calculator's intersect feature, you can conclude that the solution set is $\{x \mid x \geq 0.5\}$.

Exercises Solve each equation or inequality by graphing.

1. $3.5^{x+2} = 1.75^{x+3}$
2. $-3^{x+4} = -0.5^{2x+3}$
3. $6^{2-x} - 4 = -0.25^{x-2.5}$
4. $3^x - 4 = 5^{\frac{x}{2}}$
5. $\log_2 3x = \log_3 (2x + 2)$
6. $2^{x-2} \geq 0.5^{x-3}$
7. $\log_3 (3x - 5) \geq \log_3 (x + 7)$
8. $5^{x+3} \leq 2^{x+4}$
9. $\log_2 2x \leq \log_4 (x + 3)$

10-5 Base e and Natural Logarithms

What You'll Learn

- Evaluate expressions involving the natural base and natural logarithms.
- Solve exponential equations and inequalities using natural logarithms.

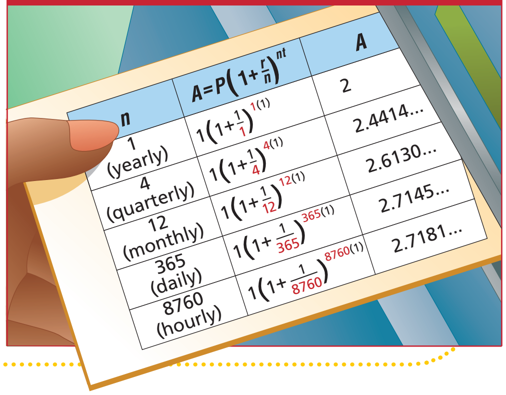
Vocabulary

- natural base, e
- natural base exponential function
- natural logarithm
- natural logarithmic function

How is the natural base e used in banking?

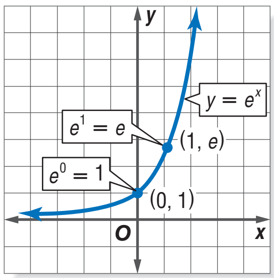
Suppose a bank compounds interest on accounts *continuously*, that is, with no waiting time between interest payments. In order to develop an equation to determine continuously compounded interest, examine what happens to the value A of an account for increasingly larger numbers of compounding periods n. Use a principal P of \$1, an interest rate r of 100% or 1, and time t of 1 year.

Continuously Compounded Interest



BASE e AND NATURAL LOGARITHMS In the table above, as n increases, the expression $1\left(1+\frac{1}{n}\right)^{n(1)}$ or $\left(1+\frac{1}{n}\right)^{n}$ approaches the irrational number 2.71828... . This number is referred to as the **natural base, e**.

An exponential function with base e is called a **natural base exponential function**. The graph of $y = e^x$ is shown at the right. Natural base exponential functions are used extensively in science to model quantities that grow and decay continuously.



Most calculators have an e^x function for evaluating natural base expressions.

Study Tip

Simplifying Expressions with e

You can simplify expressions involving e in the same manner in which you simplify expressions involving π.

Examples:

- $\pi^2 \cdot \pi^3 = \pi^5$
- $e^2 \cdot e^3 = e^5$

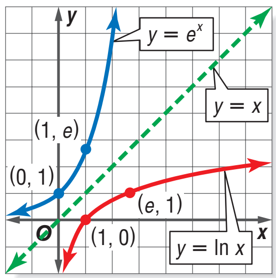
Example 1 Evaluate Natural Base Expressions

Use a calculator to evaluate each expression to four decimal places.

a. e^2 KEYSTROKES: [2nd] [e^x] 2 [ENTER] 7.389056099 about 7.3891

b. $e^{-1.3}$ KEYSTROKES: [2nd] [e^x] -1.3 [ENTER] .272531793 about 0.2725

The logarithm with base e is called the **natural logarithm**, sometimes denoted by $\log_e x$, but more often abbreviated $\ln x$. The **natural logarithmic function**, $y = \ln x$, is the inverse of the natural base exponential function, $y = e^x$. The graph of these two functions shows that $\ln 1 = 0$ and $\ln e = 1$.



Most calculators have an LN key for evaluating natural logarithms.

Example 2 Evaluate Natural Logarithmic Expressions

Use a calculator to evaluate each expression to four decimal places.

a. $\ln 4$ KEYSTROKES: LN 4 ENTER 1.386294361 about 1.3863

b. $\ln 0.05$ KEYSTROKES: LN 0.05 ENTER −2.995732274 about −2.9957

You can write an equivalent base e exponential equation for a natural logarithmic equation and vice versa by using the fact that $\ln x = \log_e x$.

Example 3 Write Equivalent Expressions

Write an equivalent exponential or logarithmic equation.

a. $e^x = 5$

$e^x = 5 \quad \rightarrow \quad \log_e 5 = x$

$\ln 5 = x$

b. $\ln x \approx 0.6931$

$\ln x \approx 0.6931 \quad \rightarrow \quad \log_e x \approx 0.6931$

$x \approx e^{0.6931}$

Since the natural base function and the natural logarithmic function are inverses, these two functions can be used to "undo" each other.

$$e^{\ln x} = x \qquad \ln e^x = x$$

Example 4 Inverse Property of Base e and Natural Logarithms

Evaluate each expression.

a. $e^{\ln 7}$

$e^{\ln 7} = 7$

b. $\ln e^{4x + 3}$

$\ln e^{4x + 3} = 4x + 3$

EQUATIONS AND INEQUALITIES WITH e AND ln Equations and inequalities involving base e are easier to solve using natural logarithms than using common logarithms. All of the properties of logarithms that you have learned apply to natural logarithms as well.

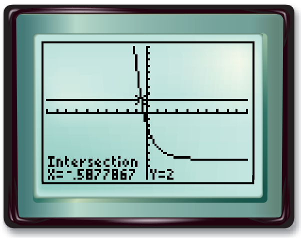
Example 5 Solve Base e Equations

Solve $5e^{-x} - 7 = 2$.

$5e^{-x} - 7 = 2$	Original equation
$5e^{-x} = 9$	Add 7 to each side.
$e^{-x} = \frac{9}{5}$	Divide each side by 5.
$\ln e^{-x} = \ln \frac{9}{5}$	Property of Equality for Logarithms
$-x = \ln \frac{9}{5}$	Inverse Property of Exponents and Logarithms
$x = -\ln \frac{9}{5}$	Divide each side by −1.
$x \approx -0.5878$	Use a calculator.

The solution is about −0.5878.

CHECK You can check this value by substituting −0.5878 into the original equation or by finding the intersection of the graphs of $y = 5e^{-x} - 7$ and $y = 2$.



Study Tip

Continuously Compounded Interest

Although no banks actually pay interest compounded continuously, the equation $A = Pe^{rt}$ is so accurate in computing the amount of money for quarterly compounding, or daily compounding, that it is often used for this purpose.

When interest is compounded continuously, the amount A in an account after t years is found using the formula $A = Pe^{rt}$, where P is the amount of principal and r is the annual interest rate.

Example 6 Solve Base *e* Inequalities

SAVINGS **Suppose you deposit \$1000 in an account paying 5% annual interest, compounded continuously.**

a. What is the balance after 10 years?

$A = Pe^{rt}$	Continuous compounding formula
$= 1000e^{(0.05)(10)}$	Replace P with 1000, r with 0.05, and t with 10.
$= 1000e^{0.5}$	Simplify.
≈ 1648.72	Use a calculator.

The balance after 10 years would be \$1648.72.

b. How long will it take for the balance in your account to reach at least \$1500?

$$\underbrace{\text{The balance}}_{A} \ \underbrace{\text{is at least}}_{\geq} \ \underbrace{\$1500.}_{1500}$$

$A \geq 1500$	Write an inequality.
$1000e^{(0.05)t} \geq 1500$	Replace A with $1000e^{(0.05)t}$.
$e^{(0.05)t} \geq 1.5$	Divide each side by 1000.
$\ln e^{(0.05)t} \geq \ln 1.5$	Property of Equality for Logarithms
$0.05t \geq \ln 1.5$	Inverse Property of Exponents and Logarithms
$t \geq \frac{\ln 1.5}{0.05}$	Divide each side by 0.05.
$t \geq 8.11$	Use a calculator.

It will take at least 8.11 years for the balance to reach \$1500.

Study Tip

Equations with ln

As with other logarithmic equations, remember to check for extraneous solutions.

Example 7 Solve Natural Log Equations and Inequalities

Solve each equation or inequality.

a. $\ln 5x = 4$

$\ln 5x = 4$	Original equation
$e^{\ln 5x} = e^4$	Write each side using exponents and base e.
$5x = e^4$	Inverse Property of Exponents and Logarithms
$x = \frac{e^4}{5}$	Divide each side by 5.
$x \approx 10.9196$	Use a calculator.

The solution is 10.9196. Check this solution using substitution or graphing.

b. $\ln (x - 1) > -2$

$\ln (x - 1) > -2$	Original inequality
$e^{\ln (x-1)} > e^{-2}$	Write each side using exponents and base e.
$x - 1 > e^{-2}$	Inverse Property of Exponents and Logarithms
$x > e^{-2} + 1$	Add 1 to each side.
$x > 1.1353$	Use a calculator.

The solution is all numbers greater than about 1.1353. Check this solution using substitution.

Check for Understanding

Concept Check

1. **Name** the base of natural logarithms.
2. **OPEN ENDED** Give an example of an exponential equation that requires using natural logarithms instead of common logarithms to solve.
3. **FIND THE ERROR** Colby and Elsu are solving $\ln 4x = 5$.

Colby	Elsu
$\ln 4x = 5$	$\ln 4x = 5$
$10^{\ln 4x} = 10^5$	$e^{\ln 4x} = e^5$
$4x = 100{,}000$	$4x = e^5$
$x = 25{,}000$	$x = \frac{e^5}{4}$
	$x \approx 37.1033$

Who is correct? Explain your reasoning.

Guided Practice

Use a calculator to evaluate each expression to four decimal places.

4. e^6
5. $e^{-3.4}$
6. $\ln 1.2$
7. $\ln 0.1$

Write an equivalent exponential or logarithmic equation.

8. $e^x = 4$
9. $\ln 1 = 0$

Evaluate each expression.

10. $e^{\ln 3}$
11. $\ln e^{5x}$

Solve each equation or inequality.

12. $e^x > 30$
13. $2e^x - 5 = 1$
14. $3 + e^{-2x} = 8$
15. $\ln x < 6$
16. $2 \ln 3x + 1 = 5$
17. $\ln x^2 = 9$

Application

ALTITUDE **For Exercises 18 and 19, use the following information.**
The altimeter in an airplane gives the altitude or height h (in feet) of a plane above sea level by measuring the outside air pressure P (in kilopascals). The height and air pressure are related by the model $P = 101.3\, e^{-\frac{h}{26{,}200}}$.

18. Find a formula for the height in terms of the outside air pressure.
19. Use the formula you found in Exercise 18 to approximate the height of a plane above sea level when the outside air pressure is 57 kilopascals.

Practice and Apply

Homework Help

For Exercises	See Examples
20–29	1, 2
30–33	3
34–37	4
38–53	5–7
54–57	6
58–61	3, 5

Extra Practice
See page 850.

Use a calculator to evaluate each expression to four decimal places.

20. e^4
21. e^5
22. $e^{-1.2}$
23. $e^{0.5}$
24. $\ln 3$
25. $\ln 10$
26. $\ln 5.42$
27. $\ln 0.03$

28. **SAVINGS** If you deposit \$150 in a savings account paying 4% interest compounded continuously, how much money will you have after 5 years? Use the formula presented in Example 6.

29. **PHYSICS** The equation $\ln \frac{I_0}{I} = 0.014d$ relates the intensity of light at a depth of d centimeters of water I with the intensity in the atmosphere I_0. Find the depth of the water where the intensity of light is half the intensity of the light in the atmosphere.

Write an equivalent exponential or logarithmic equation.

30. $e^{-x} = 5$ **31.** $e^2 = 6x$ **32.** $\ln e = 1$ **33.** $\ln 5.2 = x$

Evaluate each expression.

34. $e^{\ln 0.2}$ **35.** $e^{\ln y}$ **36.** $\ln e^{-4x}$ **37.** $\ln e^{45}$

Solve each equation or inequality.

38. $3e^x + 1 = 5$ **39.** $2e^x - 1 = 0$ **40.** $e^x < 4.5$

41. $e^x > 1.6$ **42.** $-3e^{4x} + 11 = 2$ **43.** $8 + 3e^{3x} = 26$

44. $e^{5x} \geq 25$ **45.** $e^{-2x} \leq 7$ **46.** $\ln 2x = 4$

47. $\ln 3x = 5$ **48.** $\ln (x + 1) = 1$ **49.** $\ln (x - 7) = 2$

50. $\ln x + \ln 3x = 12$ **51.** $\ln 4x + \ln x = 9$

52. $\ln (x^2 + 12) = \ln x + \ln 8$ **53.** $\ln x + \ln (x + 4) = \ln 5$

More About. . .

Money

To determine the doubling time on an account paying an interest rate r that is compounded *annually*, investors use the "Rule of 72." Thus, the amount of time needed for the money in an account paying 6% interest compounded annually to double is $\frac{72}{6}$ or 12 years.

Source: www.datachimp.com

MONEY **For Exercises 54–57, use the formula for continuously compounded interest found in Example 6.**

54. If you deposit \$100 in an account paying 3.5% interest compounded continuously, how long will it take for your money to double?

55. Suppose you deposit A dollars in an account paying an interest rate r as a percent, compounded continuously. Write an equation giving the time t needed for your money to double, or the *doubling time*.

56. Explain why the equation you found in Exercise 55 might be referred to as the "Rule of 70."

57. **MAKE A CONJECTURE** State a rule that could be used to approximate the amount of time t needed to triple the amount of money in a savings account paying r percent interest compounded continuously.

POPULATION **For Exercises 58 and 59, use the following information.**
In 2000, the world's population was about 6 billion. If the world's population continues to grow at a constant rate, the future population P, in billions, can be predicted by $P = 6e^{0.02t}$, where t is the time in years since 2000.

58. According to this model, what will the world's population be in 2010?

59. Some experts have estimated that the world's food supply can support a population of, at most, 18 billion. According to this model, for how many more years will the world's population remain at 18 billion or less?

Online Research **Data Update** What is the current world population? Visit www.algebra2.com/data_update to learn more.

RUMORS **For Exercises 60 and 61, use the following information.**
The number of people H who have heard a rumor can be approximated by $H = \frac{P}{1 + (P - S)e^{-0.35t}}$, where P is the total population, S is the number of people who start the rumor, and t is the time in minutes. Suppose two students start a rumor that the principal will let everyone out of school one hour early that day.

60. If there are 1600 students in the school, how many students will have heard the rumor after 10 minutes?

61. How much time will pass before half of the students have heard the rumor?

62. **CRITICAL THINKING** Determine whether the following statement is *sometimes*, *always*, or *never* true. Explain your reasoning.

For all positive numbers x and y, $\frac{\log x}{\log y} = \frac{\ln x}{\ln y}$.

63. WRITING IN MATH Answer the question that was posed at the beginning of the lesson.

How is the natural base e used in banking?

Include the following in your answer:

- an explanation of how to calculate the value of an account whose interest is compounded continuously, and
- an explanation of how to use natural logarithms to find when the account will have a specified value.

Standardized Test Practice

64. If $e^x \neq 1$ and $e^{x^2} = \frac{1}{(\sqrt{2})^x}$, what is the value of x?

(A) -1.41 (B) -0.35 (C) 1.00 (D) 1.10

65. **SHORT RESPONSE** The population of a certain country can be modeled by the equation $P(t) = 40\,e^{0.02t}$, where P is the population in millions and t is the number of years since 1900. When will the population be 100 million, 200 million, and 400 million? What do you notice about these time periods?

Maintain Your Skills

Mixed Review

Express each logarithm in terms of common logarithms. Then approximate its value to four decimal places. *(Lesson 10-4)*

66. $\log_4 68$ 67. $\log_6 0.047$ 68. $\log_{50} 23$

Solve each equation. Check your solutions. *(Lesson 10-3)*

69. $\log_3 (a + 3) + \log_3 (a - 3) = \log_3 16$ 70. $\log_{11} 2 + 2 \log_{11} x = \log_{11} 32$

State whether each equation represents a *direct*, *joint*, or *inverse* variation. Then name the constant of variation. *(Lesson 9-4)*

71. $mn = 4$ 72. $\frac{a}{b} = c$ 73. $y = -7x$

74. **COMMUNICATION** A microphone is placed at the focus of a parabolic reflector to collect sounds for the television broadcast of a football game. The focus of the parabola that is the cross section of the reflector is 5 inches from the vertex. The latus rectum is 20 inches long. Assuming that the focus is at the origin and the parabola opens to the right, write the equation of the cross section. *(Lesson 8-2)*

Getting Ready for the Next Lesson

PREREQUISITE SKILL **Solve each equation or inequality.**
*(To review **exponential equations and inequalities**, see Lesson 10-1.)*

75. $2^x = 10$ 76. $5^x = 12$ 77. $6^x = 13$

78. $2(1 + 0.1)^x = 50$ 79. $10(1 + 0.25)^x = 200$ 80. $400(1 - 0.2)^x = 50$

Practice Quiz 2

Lessons 10-3 through 10-5

1. Express $\log_4 5$ in terms of common logarithms. Then approximate its value to four decimal places. *(Lesson 10-4)*

2. Write an equivalent exponential equation for $\ln 3x = 2$. *(Lesson 10-5)*

Solve each equation or inequality. *(Lesson 10-3 through 10-5)*

3. $\log_2 (9x + 5) = 2 + \log_2 (x^2 - 1)$ 4. $2^{x-3} > 5$ 5. $2e^x - 1 = 7$

10-6 Exponential Growth and Decay

What You'll Learn

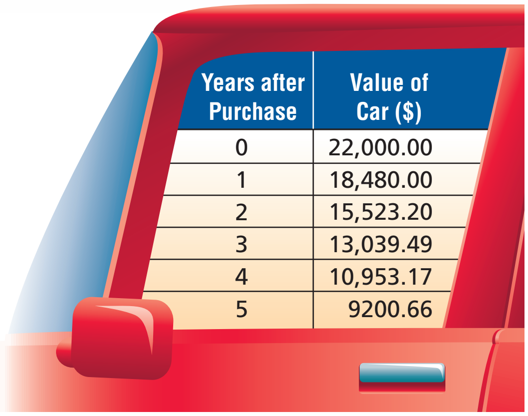
- Use logarithms to solve problems involving exponential decay.
- Use logarithms to solve problems involving exponential growth.

Vocabulary

- rate of decay
- rate of growth

How can you determine the current value of your car?

Certain assets, like homes, can *appreciate* or increase in value over time. Others, like cars, *depreciate* or decrease in value with time. Suppose you buy a car for \$22,000 and the value of the car decreases by 16% each year. The table shows the value of the car each year for up to 5 years after it was purchased.



Years after Purchase	Value of Car (\$)
0	22,000.00
1	18,480.00
2	15,523.20
3	13,039.49
4	10,953.17
5	9200.66

EXPONENTIAL DECAY The depreciation of the value of a car is an example of exponential decay. When a quantity *decreases* by a fixed percent each year, or other period of time, the amount y of that quantity after t years is given by $y = a(1 - r)^t$, where a is the initial amount and r is the percent of decrease expressed as a decimal. The percent of decrease r is also referred to as the **rate of decay**.

Example 1 Exponential Decay of the Form $y = a(1 - r)^t$

CAFFEINE **A cup of coffee contains 130 milligrams of caffeine. If caffeine is eliminated from the body at a rate of 11% per hour, how long will it take for half of this caffeine to be eliminated from a person's body?**

Study Tip

Rate of Change
Remember to rewrite the rate of change as a decimal before using it in the formula.

Explore The problem gives the amount of caffeine consumed and the rate at which the caffeine is eliminated. It asks you to find the time it will take for half of the caffeine to be eliminated from a person's body.

Plan Use the formula $y = a(1 - r)^t$. Let t be the number of hours since drinking the coffee. The amount remaining y is half of 130 or 65.

Solve

$y = a(1 - r)^t$ Exponential decay formula

$65 = 130(1 - 0.11)^t$ Replace y with 65, a with 130, and r with 11% or 0.11.

$0.5 = (0.89)^t$ Divide each side by 130.

$\log 0.5 = \log (0.89)^t$ Property of Equality for Logarithms

$\log 0.5 = t \log (0.89)$ Product Property for Logarithms

$\frac{\log 0.5}{\log 0.89} = t$ Divide each side by log 0.89.

$5.9480 \approx t$ Use a calculator.

It will take approximately 6 hours for half of the caffeine to be eliminated from a person's body.

Examine Use the formula to find how much of the original 130 milligrams of caffeine would remain after 6 hours.

$y = a(1 - r)^t$	Exponential decay formula
$y = 130(1 - 0.11)^6$	Replace *a* with 130, *r* with 0.11, and *t* with 6.
$y \approx 64.6$	Use a calculator.

Half of 130 is 65, so the answer seems reasonable.

Another model for exponential decay is given by $y = ae^{-kt}$, where k is a constant. This is the model preferred by scientists. Use this model to solve problems involving radioactive decay.

Example 2 Exponential Decay of the Form $y = ae^{-kt}$

PALEONTOLOGY **The *half-life* of a radioactive substance is the time it takes for half of the atoms of the substance to become disintegrated. All life on Earth contains the radioactive element Carbon-14, which decays continuously at a fixed rate. The half-life of Carbon-14 is 5760 years. That is, every 5760 years half of a mass of Carbon-14 decays away.**

a. What is the value of k for Carbon-14?

To determine the constant k for Carbon-14, let a be the initial amount of the substance. The amount y that remains after 5760 years is then represented by $\frac{1}{2}a$ or $0.5a$.

$y = ae^{-kt}$	Exponential decay formula
$0.5a = ae^{-k(5760)}$	Replace *y* with 0.5*a* and *t* with 5760.
$0.5 = e^{-5760k}$	Divide each side by *a*.
$\ln 0.5 = \ln e^{-5760k}$	Property of Equality for Logarithmic Functions
$\ln 0.5 = -5760k$	Inverse Property of Exponents and Logarithms
$\frac{\ln 0.5}{-5760} = k$	Divide each side by −5760.
$0.00012 \approx k$	Use a calculator.

The constant for Carbon-14 is 0.00012. Thus, the equation for the decay of Carbon-14 is $y = ae^{-0.00012t}$, where t is given in years.

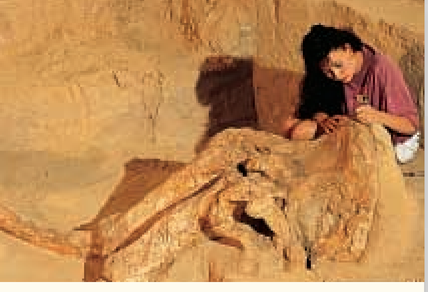
b. A paleontologist examining the bones of a woolly mammoth estimates that they contain only 3% as much Carbon-14 as they would have contained when the animal was alive. How long ago did the mammoth die?

Let a be the initial amount of Carbon-14 in the animal's body. Then the amount y that remains after t years is 3% of a or $0.03a$.

$y = ae^{-0.00012t}$	Formula for the decay of Carbon-14
$0.03a = ae^{-0.00012t}$	Replace *y* with 0.03*a*.
$0.03 = e^{-0.00012t}$	Divide each side by *a*.
$\ln 0.03 = \ln e^{-0.00012t}$	Property of Equality for Logarithms
$\ln 0.03 = -0.00012t$	Inverse Property of Exponents and Logarithms
$\frac{\ln 0.03}{-0.00012} = t$	Divide each side by −0.00012.
$29{,}221 \approx t$	Use a calculator.

The mammoth lived about 29,000 years ago.

Career Choices



Paleontologist

Paleontologists study fossils found in geological formations. They use these fossils to trace the evolution of plant and animal life and the geologic history of Earth.

Online Research For information about a career as a paleontologist, visit: www.algebra2.com/careers

Source: U.S. Department of Labor

EXPONENTIAL GROWTH When a quantity *increases* by a fixed percent each time period, the amount y of that quantity after t time periods is given by $y = a(1 + r)^t$, where a is the initial amount and r is the percent of increase expressed as a decimal. The percent of increase r is also referred to as the **rate of growth**.

Standardized Test Practice

Example 3 Exponential Growth of the Form $y = a(1 + r)^t$

Multiple-Choice Test Item

In 1910, the population of a city was 120,000. Since then, the population has increased by exactly 1.5% per year. If the population continues to grow at this rate, what will the population be in 2010?

(A) 138,000	(B) 531,845
(C) 1,063,690	(D) 1.4×10^{11}

The Princeton Review

Test-Taking Tip

To change a percent to a decimal, drop the percent symbol and move the decimal point two places to the left.

$1.5\% = 0.015$

Read the Test Item

You need to find the population of the city 2010 − 1910 or 100 years later. Since the population is growing at a fixed percent each year, use the formula $y = a(1 + r)^t$.

Solve the Test Item

$y = a(1 + r)^t$ Exponential growth formula

$y = 120{,}000(1 + 0.015)^{100}$ Replace $a = 120{,}000$, r with 0.015, and t with 2010 − 1910 or 100.

$y = 120{,}000(1.015)^{100}$ Simplify.

$y \approx 531{,}845.48$ Use a calculator.

The answer is B.

Another model for exponential growth, preferred by scientists, is $y = ae^{kt}$, where k is a constant. Use this model to find the constant k.

Example 4 Exponential Growth of the Form $y = ae^{kt}$

POPULATION **As of 2000, China was the world's most populous country, with an estimated population of 1.26 billion people. The second most populous country was India, with 1.01 billion. The populations of India and China can be modeled by $I(t) = 1.01e^{0.015t}$ and $C(t) = 1.26e^{0.009t}$, respectively. According to these models, when will India's population be more than China's?**

You want to find t such that $I(t) > C(t)$.

$I(t) > C(t)$

$1.01e^{0.015t} > 1.26e^{0.009t}$ Replace $I(t)$ with $1.01e^{0.015t}$ and $C(t)$ with $1.26e^{0.009t}$

$\ln 1.01e^{0.015t} > \ln 1.26e^{0.009t}$ Property of Inequality for Logarithms

$\ln 1.01 + \ln e^{0.015t} > \ln 1.26 + \ln e^{0.009t}$ Product Property of Logarithms

$\ln 1.01 + 0.015t > \ln 1.26 + 0.009t$ Inverse Property of Exponents and Logarithms

$0.006t > \ln 1.26 - \ln 1.01$ Subtract 0.009 from each side.

$t > \dfrac{\ln 1.26 - \ln 1.01}{0.006}$ Divide each side by 0.006.

$t > 36.86$ Use a calculator.

After 37 years or in 2037, India will be the most populous country in the world.

Check for Understanding

Concept Check

1. **Write** a general formula for exponential growth and decay where r is the percent of change.
2. **Explain** how to solve $y = (1 + r)^t$ for t.
3. **OPEN ENDED** Give an example of a quantity that grows or decays at a fixed rate.

Guided Practice

SPACE For Exercises 4–6, use the following information.
A radioisotope is used as a power source for a satellite. The power output P (in watts) is given by $P = 50e^{-\frac{t}{250}}$, where t is the time in days.

4. Is the formula for power output an example of exponential growth or decay? Explain your reasoning.
5. Find the power available after 100 days.
6. Ten watts of power are required to operate the equipment in the satellite. How long can the satellite continue to operate?

POPULATION GROWTH For Exercises 7 and 8, use the following information.
The city of Raleigh, North Carolina, grew from a population of 212,000 in 1990 to a population of 259,000 in 1998.

7. Write an exponential growth equation of the form $y = ae^{kt}$ for Raleigh, where t is the number of years after 1990.
8. Use your equation to predict the population of Raleigh in 2010.

Standardized Test Practice

A B C D

9. Suppose the weight of a bar of soap decreases by 2.5% each time it is used. If the bar weighs 95 grams when it is new, what is its weight to the nearest gram after 15 uses?

   (A) 57.5 g (B) 59.4 g (C) 65 g (D) 93 g

Practice and Apply

Homework Help

For Exercises	See Examples
10	1
12–14,	2
11, 17–20	3
15, 16	4

Extra Practice

See page 851.

10. **COMPUTERS** Zeus Industries bought a computer for \$2500. It is expected to depreciate at a rate of 20% per year. What will the value of the computer be in 2 years?

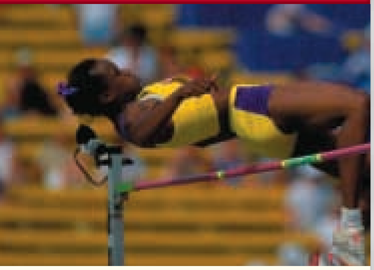
11. **REAL ESTATE** The Martins bought a condominium for \$85,000. Assuming that the value of the condo will appreciate at most 5% a year, how much will the condo be worth in 5 years?

12. **MEDICINE** Radioactive iodine is used to determine the health of the thyroid gland. It decays according to the equation $y = ae^{-0.0856t}$, where t is in days. Find the half-life of this substance.

13. **PALEONTOLOGY** A paleontologist finds a bone that might be a dinosaur bone. In the laboratory, she finds that the Carbon-14 found in the bone is $\frac{1}{12}$ of that found in living bone tissue. Could this bone have belonged to a dinosaur? Explain your reasoning. (*Hint:* The dinosaurs lived from 220 million years ago to 63 million years ago.)

14. **ANTHROPOLOGY** An anthropologist finds there is so little remaining Carbon-14 in a prehistoric bone that instruments cannot measure it. This means that there is less than 0.5% of the amount of Carbon-14 the bones would have contained when the person was alive. How long ago did the person die?

More About. . .



Olympics

The women's high jump competition first took place in the USA in 1895, but it did not become an Olympic event until 1928.

Source: www.princeton.edu

BIOLOGY **For Exercises 15 and 16, use the following information.**
Bacteria usually reproduce by a process known as *binary fission*. In this type of reproduction, one bacterium divides, forming two bacteria. Under ideal conditions, some bacteria reproduce every 20 minutes.

15. Find the constant k for this type of bacteria under ideal conditions.

16. Write the equation for modeling the exponential growth of this bacterium.

ECONOMICS **For Exercises 17 and 18, use the following information.**
The annual Gross Domestic Product (GDP) of a country is the value of all of the goods and services produced in the country during a year. During the period 1985–1999, the Gross Domestic Product of the United States grew about 3.2% per year, measured in 1996 dollars. In 1985, the GDP was $5717 billion.

17. Assuming this rate of growth continues, what will the GDP of the United States be in the year 2010?

18. In what year will the GDP reach $20 trillion?

19. **OLYMPICS** In 1928, when the high jump was first introduced as a women's sport at the Olympic Games, the winning women's jump was 62.5 inches, while the winning men's jump was 76.5 inches. Since then, the winning jump for women has increased by about 0.38% per year, while the winning jump for men has increased at a slower rate, 0.3%. If these rates continue, when will the women's winning high jump be higher than the men's?

20. **HOME OWNERSHIP** The Mendes family bought a new house 10 years ago for $120,000. The house is now worth $191,000. Assuming a steady rate of growth, what was the yearly rate of appreciation?

21. **CRITICAL THINKING** The half-life of Radium is 1620 years. When will a 20-gram sample of Radium be completely gone? Explain your reasoning.

22. **WRITING IN MATH** Answer the question that was posed at the beginning of the lesson.

How can you determine the current value of your car?

Include the following in your answer:
- a description of how to find the percent decrease in the value of the car each year, and
- a description of how to find the value of a car for any given year when the rate of depreciation is known.

Standardized Test Practice

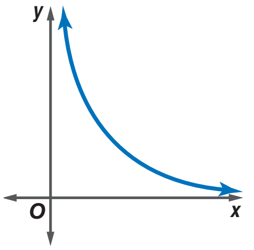
A B C D

23. **SHORT RESPONSE** An artist creates a sculpture out of salt that weighs 2000 pounds. If the sculpture loses 3.5% of its mass each year to erosion, after how many years will the statue weigh less than 1000 pounds?

24. The curve shown at the right represents a portion of the graph of which function?

(A) $y = 50 - x$ (B) $y = \log x$

(C) $y = e^{-x}$ (D) $xy = 5$



Maintain Your Skills

Mixed Review

Write an equivalent exponential or logarithmic equation. *(Lesson 10-5)*

25. $e^3 = y$ **26.** $e^{4n-2} = 29$ **27.** $\ln 4 + 2 \ln x = 8$

Solve each equation or inequality. Round to four decimal places. *(Lesson 10-4)*

28. $16^x = 70$ **29.** $2^{3p} > 1000$ **30.** $\log_b 81 = 2$

BUSINESS **For Exercises 31–33, use the following information.**
The board of a small corporation decided that 8% of the annual profits would be divided among the six managers of the corporation. There are two sales managers and four nonsales managers. Fifty percent of the amount would be split equally among all six managers. The other 50% would be split among the four nonsales managers. Let p represent the annual profits of the corporation. *(Lesson 9-2)*

31. Write an expression to represent the share of the profits each nonsales manager will receive.

32. Simplify this expression.

33. Write an expression in simplest form to represent the share of the profits each sales manager will receive.

Without writing the equation in standard form, state whether the graph of each equation is a *parabola, circle, ellipse,* or *hyperbola*. *(Lesson 8-6)*

34. $4y^2 - 3x^2 + 8y - 24x = 50$ **35.** $7x^2 - 42x + 6y^2 - 24y = -45$

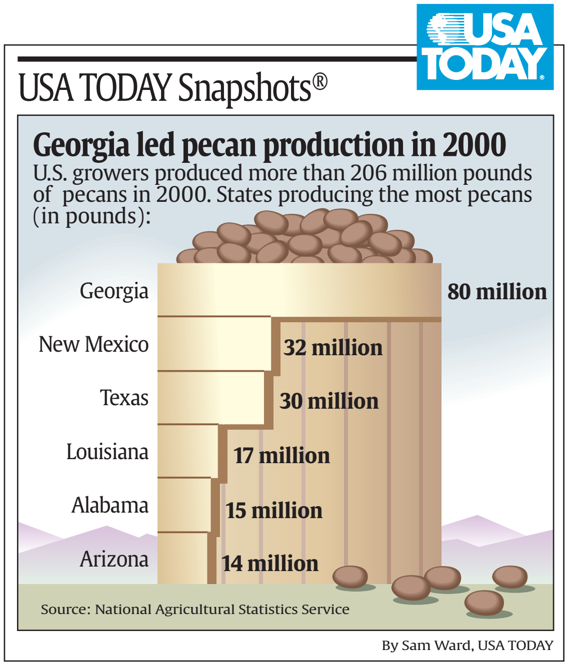
36. $y^2 + 3x - 8y = 4$ **37.** $x^2 + y^2 - 6x + 2y + 5 = 0$

AGRICULTURE **For Exercises 38–40, use the graph at the right.** *(Lesson 5-1)*

38. Write the number of pounds of pecans produced by U.S. growers in 2000 in scientific notation.

39. Write the number of pounds of pecans produced by the state of Georgia in 2000 in scientific notation.

40. What percent of the overall pecan production for 2000 can be attributed to Georgia?



WebQuest Internet Project

On Quake Anniversary, Japan Still Worries

It is time to complete your project. Use the information and data you have gathered about earthquakes to prepare a research report or Web page. Be sure to include graphs, tables, diagrams, and any calculations you need for the earthquake you chose.

www.algebra2.com/webquest

Chapter 10 Study Guide and Review

Vocabulary and Concept Check

Change of Base Formula (p. 548)
common logarithm (p. 547)
exponential decay (p. 524)
exponential equation (p. 526)
exponential function (p. 524)
exponential growth (p. 524)
exponential inequality (p. 527)
logarithm (p. 531)
logarithmic equation (p. 533)
logarithmic function (p. 532)
logarithmic inequality (p. 533)
natural base, e (p. 554)
natural base exponential function (p. 554)
natural logarithm (p. 554)
natural logarithmic function (p. 554)
Power Property of Logarithms (p. 543)
Product Property of Logarithms (p. 541)
Property of Equality for Exponential Functions (p. 526)
Property of Equality for Logarithmic Functions (p. 534)
Property of Inequality for Exponential Functions (p. 527)
Property of Inequality for Logarithmic Functions (p. 534)
Quotient Property of Logarithms (p. 542)
rate of decay (p. 560)
rate of growth (p. 562)

State whether each sentence is *true* or *false*. If false, replace the underlined word(s) to make a true statement.

1. If $24^{2y+3} = 24^{y-4}$, then $2y + 3 = y - 4$ by the *Property of Equality for Exponential Functions*.
2. The number of bacteria in a petri dish over time is an example of *exponential decay*.
3. The *natural logarithm* is the inverse of the exponential function with base 10.
4. The *Power Property of Logarithms* shows that $\ln 9 < \ln 81$.
5. If a savings account yields 2% interest per year, then 2% is the *rate of growth*.
6. Radioactive half-life is an example of *exponential decay*.
7. The inverse of an exponential function is a *composite function*.
8. The *Quotient Property of Logarithms* is shown by $\log_4 2x = \log_4 2 + \log_4 x$.
9. The function $f(x) = 2(5)^x$ is an example of a *quadratic function*.

Lesson-by-Lesson Review

10-1 Exponential Functions

See pages 523–530.

Concept Summary

- An exponential function is in the form $y = ab^x$, where $a \neq 0$, $b > 0$, and $b \neq 1$.
- The function $y = ab^x$ represents exponential growth for $a > 0$ and $b > 1$, and exponential decay for $a > 0$ and $0 < b < 1$.
- Property of Equality for Exponential Functions: If b is a positive number other than 1, then $b^x = b^y$ if and only if $x = y$.
- Property of Inequality for Exponential Functions: If $b > 1$, then $b^x > b^y$ if and only if $x > y$, and $b^x < b^y$ if and only if $x < y$.

Example **Solve $64 = 2^{3n+1}$ for n.**

$64 = 2^{3n+1}$ Original equation

$2^6 = 2^{3n+1}$ Rewrite 64 as 2^6 so each side has the same base.

$6 = 3n + 1$ Property of Equality for Exponential Functions

$\frac{5}{3} = n$ The solution is $\frac{5}{3}$.

Exercises **Determine whether each function represents exponential *growth* or *decay*.** *See Example 2 on page 525.*

10. $y = 5(0.7)^x$

11. $y = \frac{1}{3}(4)^x$

Write an exponential function whose graph passes through the given points. *See Example 3 on page 525.*

12. $(0, -2)$ and $(3, -54)$

13. $(0, 7)$ and $(1, 1.4)$

Solve each equation or inequality. *See Examples 5 and 6 on pages 526 and 527.*

14. $9^x = \frac{1}{81}$

15. $2^{6x} = 4^{5x+2}$

16. $49^{3p+1} = 7^{2p-5}$

17. $9^{x^2} \leq 27^{x^2-2}$

10-2 Logarithms and Logarithmic Functions

See pages 531–538.

Concept Summary

- Suppose $b > 0$ and $b \neq 1$. For $x > 0$, there is a number y such that $\log_b x = y$ if and only if $b^y = x$.
- Logarithmic to exponential inequality:
  If $b > 1$, $x > 0$, and $\log_b x > y$, then $x > b^y$.

  If $b > 1$, $x > 0$, and $\log_b x < y$, then $0 < x < b^y$.
- Property of Equality for Logarithmic Functions:
  If b is a positive number other than 1, then $\log_b x = \log_b y$ if and only if $x = y$.
- Property of Inequality for Logarithmic Functions:
  If $b > 1$, then $\log_b x > \log_b y$ if and only if $x > y$, and $\log_b x < \log_b y$ if and only if $x < y$.

Examples

$\log_7 x = 2 \rightarrow 7^2 = x$

$\log_2 x > 5 \rightarrow x > 2^5$

$\log_3 x < 4 \rightarrow 0 < x < 3^4$

If $\log_5 x = \log_5 6$, then $x = 6$.

If $\log_4 x > \log_4 10$, then $x > 10$.

Examples **1** **Solve $\log_9 n > \frac{3}{2}$.**

$\log_9 n > \frac{3}{2}$ Original inequality

$n > 9^{\frac{3}{2}}$ Logarithmic to exponential inequality

$n > (3^2)^{\frac{3}{2}}$ $9 = 3^2$

$n > 3^3$ Power of a Power

$n > 27$ Simplify.

2 **Solve $\log_3 12 = \log_3 2x$.**

$\log_3 12 = \log_3 2x$ Original equation

$12 = 2x$ Property of Equality for Logarithmic Functions

$6 = x$ Divide each side by 2.

Exercises **Write each equation in logarithmic form.** *See Example 1 on page 532.*

18. $7^3 = 343$ **19.** $5^{-2} = \frac{1}{25}$ **20.** $4^{\frac{3}{2}} = 8$

Write each equation in exponential form. *See Example 2 on page 532.*

21. $\log_4 64 = 3$ **22.** $\log_8 2 = \frac{1}{3}$ **23.** $\log_6 \frac{1}{36} = -2$

Evaluate each expression. *See Examples 3 and 4 on pages 532 and 533.*

24. $4^{\log_4 9}$ **25.** $\log_7 7^{-5}$ **26.** $\log_{81} 3$ **27.** $\log_{13} 169$

Solve each equation or inequality. *See Examples 5–8 on pages 533 and 534.*

28. $\log_4 x = \frac{1}{2}$ **29.** $\log_{81} 729 = x$

30. $\log_b 9 = 2$ **31.** $\log_8 (3y - 1) < \log_8 (y + 5)$

32. $\log_5 12 < \log_5 (5x - 3)$ **33.** $\log_8 (x^2 + x) = \log_8 12$

10-3 Properties of Logarithms

See pages 541–546.

Concept Summary

- The logarithm of a product is the sum of the logarithms of its factors.
- The logarithm of a quotient is the difference of the logarithms of the numerator and the denominator.
- The logarithm of a power is the product of the logarithm and the exponent.

Example **Use $\log_{12} 9 \approx 0.884$ and $\log_{12} 18 \approx 1.163$ to approximate the value of $\log_{12} 2$.**

$\log_{12} 2 = \log_{12} \frac{18}{9}$ Replace 2 with $\frac{18}{9}$.

$= \log_{12} 18 - \log_{12} 9$ Quotient Property

$\approx 1.163 - 0.884$ or 0.279 Replace $\log_{12} 9$ with 0.884 and $\log_{12} 18$ with 1.163.

Exercises **Use $\log_9 7 \approx 0.8856$ and $\log_9 4 \approx 0.6309$ to approximate the value of each expression.** *See Examples 1 and 2 on page 542.*

34. $\log_9 28$ **35.** $\log_9 49$ **36.** $\log_9 144$

Solve each equation. *See Example 5 on page 543.*

37. $\log_2 y = \frac{1}{3} \log_2 27$ **38.** $\log_5 7 + \frac{1}{2} \log_5 4 = \log_5 x$

39. $2 \log_2 x - \log_2 (x + 3) = 2$ **40.** $\log_3 x - \log_3 4 = \log_3 12$

41. $\log_6 48 - \log_6 \frac{16}{5} + \log_6 5 = \log_6 5x$ **42.** $\log_7 m = \frac{1}{3} \log_7 64 + \frac{1}{2} \log_7 121$

10-4 Common Logarithms

See pages 547–551.

Concept Summary

- Base 10 logarithms are called common logarithms and are usually written without the subscript 10: $\log_{10} x = \log x$.
- You use the inverse of logarithms, or exponentiation, to solve equations or inequalities involving common logarithms: $10^{\log x} = x$.
- The Change of Base Formula: $\log_a n = \dfrac{\log_b n}{\log_b a}$ ← log base b original number ← log base b old base

Example **Solve $5^x = 7$.**

$5^x = 7$	Original equation
$\log 5^x = \log 7$	Property of Equality for Logarithmic Functions
$x \log 5 = \log 7$	Power Property of Logarithms
$x = \dfrac{\log 7}{\log 5}$	Divide each side by log 5.
$x \approx \dfrac{0.8451}{0.6990}$ or 1.2090	Use a calculator.

Exercises **Solve each equation or inequality. Round to four decimal places.**
See Examples 3 and 4 on page 548.

43. $2^x = 53$ **44.** $2.3^{x^2} = 66.6$ **45.** $3^{4x-7} < 4^{2x+3}$

46. $6^{3y} = 8^{y-1}$ **47.** $12^{x-5} \geq 9.32$ **48.** $2.1^{x-5} = 9.32$

Express each logarithm in terms of common logarithms. Then approximate its value to four decimal places. *See Example 5 on page 549.*

49. $\log_4 11$ **50.** $\log_2 15$ **51.** $\log_{20} 1000$

10-5 Base e and Natural Logarithms

See pages 554–559.

Concept Summary

- You can write an equivalent base e exponential equation for a natural logarithmic equation and vice versa by using the fact that $\ln x = \log_e x$.
- Since the natural base function and the natural logarithmic function are inverses, these two functions can be used to "undo" each other.
  $e^{\ln x} = x$ and $\ln e^x = x$

Example **Solve $\ln (x + 4) > 5$.**

$\ln (x + 4) > 5$	Original inequality
$e^{\ln (x+4)} > e^5$	Write each side using exponents and base e.
$x + 4 > e^5$	Inverse Property of Exponents and Logarithms
$x > e^5 - 4$	Subtract 4 from each side.
$x > 144.4132$	Use a calculator.

Chapter 10 For More ...

- Extra Practice, see pages 849–851.
- Mixed Problem Solving, see page 871.

Exercises **Write an equivalent exponential or logarithmic equation.**
See Example 3 on page 555.

52. $e^x = 6$

53. $\ln 7.4 = x$

Evaluate each expression. *See Example 4 on page 555.*

54. $e^{\ln 12}$

55. $\ln e^{7x}$

Solve each equation or inequality.
See Examples 5 and 7 on pages 555 and 556.

56. $2e^x - 4 = 1$

57. $e^x > 3.2$

58. $-4e^{2x} + 15 = 7$

59. $\ln 3x \leq 5$

60. $\ln (x - 10) = 0.5$

61. $\ln x + \ln 4x = 10$

10-6 Exponential Growth and Decay

See pages 560–565.

Concept Summary

- Exponential decay: $y = a(1 - r)^t$ or $y = ae^{-kt}$
- Exponential growth: $y = a(1 + r)^t$ or $y = ae^{kt}$

Example **BIOLOGY** **A certain culture of bacteria will grow from 500 to 4000 bacteria in 1.5 hours. Find the constant k for the growth formula. Use $y = ne^{kt}$.**

$y = ae^{kt}$	Exponential growth formula
$4000 = 500e^{k(1.5)}$	Replace y with 4000, a with 500, and t with 1.5.
$8 = e^{1.5k}$	Divide each side by 500.
$\ln 8 = \ln e^{1.5k}$	Property of Equality for Logarithmic Functions
$\ln 8 = 1.5k$	Inverse Property of Exponents and Logarithms
$\frac{\ln 8}{1.5} = k$	Divide each side by 1.5.
$1.3863 \approx k$	Use a calculator.

Exercises *See Examples 1–4 on pages 560–562.*

62. **BUSINESS** Able Industries bought a fax machine for \$250. It is expected to depreciate at a rate of 25% per year. What will be the value of the fax machine in 3 years?

63. **BIOLOGY** For a certain strain of bacteria, k is 0.872 when t is measured in days. How long will it take 9 bacteria to increase to 738 bacteria?

64. **CHEMISTRY** Radium-226 decomposes radioactively. Its half-life, the time it takes for half of the sample to decompose, is 1800 years. Find the constant k in the decay formula for this compound.

65. **POPULATION** The population of a city 10 years ago was 45,600. Since then, the population has increased at a steady rate each year. If the population is currently 64,800, find the annual rate of growth for this city.

Chapter 10 Practice Test

Vocabulary and Concepts

Choose the term that best completes each sentence.

1. The equation $y = 0.3(4)^x$ is an exponential (*growth, decay*) function.
2. The logarithm of a quotient is the (*sum, difference*) of the logarithms of the numerator and the denominator.
3. The base of a natural logarithm is (*10, e*) .

Skills and Applications

4. Write $3^7 = 2187$ in logarithmic form.
5. Write $\log_8 16 = \frac{4}{3}$ in exponential form.
6. Write an exponential function whose graph passes through (0, 0.4) and (2, 6.4).
7. Express $\log_3 5$ in terms of common logarithms.
8. Evaluate $\log_2 \frac{1}{32}$.

Use $\log_4 7 \approx 1.4037$ and $\log_4 3 \approx 0.7925$ to approximate the value of each expression.

9. $\log_4 21$
10. $\log_4 \frac{7}{12}$

Simplify each expression.

11. $\left(3^{\sqrt{8}}\right)^{\sqrt{2}}$
12. $81^{\sqrt{5}} \div 3^{\sqrt{5}}$

Solve each equation or inequality. Round to four decimal places if necessary.

13. $2^{x-3} = \frac{1}{16}$
14. $27^{2p+1} = 3^{4p-1}$
15. $\log_2 x < 7$
16. $\log_m 144 = -2$
17. $\log_3 x - 2\log_3 2 = 3\log_3 3$
18. $\log_9 (x + 4) + \log_9 (x - 4) = 1$
19. $\log_5 (8y - 7) = \log_5 (y^2 + 5)$
20. $\log_3 3^{(4x-1)} = 15$
21. $7.6^{x-1} = 431$
22. $\log_2 5 + \frac{1}{3}\log_2 27 = \log_2 x$
23. $3^x = 5^{x-1}$
24. $4^{2x-3} = 9^{x+3}$
25. $e^{3y} > 6$
26. $2e^{3x} + 5 = 11$
27. $\ln 3x - \ln 15 = 2$

COINS For Exercises 28 and 29, use the following information.
You buy a commemorative coin for \$25. The value of the coin increases 3.25% per year.

28. How much will the coin be worth in 15 years?
29. After how many years will the coin have doubled in value?

30. **QUANTITATIVE COMPARISION Compare the quantity in Column A and the quantity in Column B. Then determine whether:**

    Ⓐ **the quantity in Column A is greater,**

    Ⓑ **the quantity in Column B is greater,**

    Ⓒ **the two quantities are equal, or**

    Ⓓ **the relationship cannot be determined from the information given.**

Column A	Column B
\$100 was deposited in an account 5 years ago.	
the current value of the account if the annual interest rate is 3% compunded quarterly	the current value of the account if the annual interest rate is 3% compounded continuously

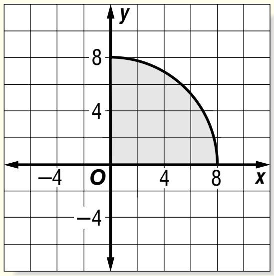
www.algebra2.com/chapter_test

Chapter 10 Standardized Test Practice

Part 1 Multiple Choice

Record your answers on the answer sheet provided by your teacher or on a sheet of paper.

1. The arc shown is part of a circle. Find the area of the shaded region.



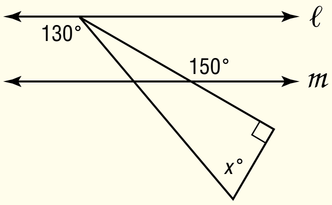
Ⓐ 8π units^2

Ⓑ 16π units^2

Ⓒ 32π units^2

Ⓓ 64π units^2

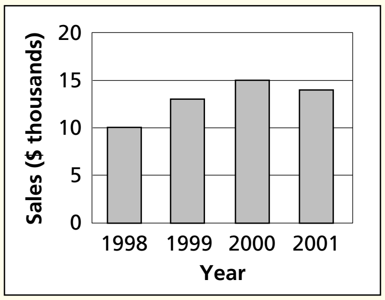
2. If line ℓ is parallel to line m in the figure below, what is the value of x?



Ⓐ 40 Ⓑ 50

Ⓒ 60 Ⓓ 70

3. According to the graph, what was the percent of increase in sales from 1998 to 2000?



Ⓐ 5% Ⓑ 15% Ⓒ 25% Ⓓ 50%

4. What is the x-intercept of the line described by the equation $y = 2x + 5$?

Ⓐ -5 Ⓑ $-\frac{5}{2}$

Ⓒ 0 Ⓓ $\frac{5}{2}$

5. $\frac{(xy)^2z^0}{y^2x^3} =$

Ⓐ $\frac{1}{x^2y}$ Ⓑ $\frac{z}{x^2}$

Ⓒ $\frac{z}{x}$ Ⓓ $\frac{1}{x}$

6. If $\frac{v^2 - 36}{6 - v} = 10$, then $v =$

Ⓐ $-16.$ Ⓑ $-4.$

Ⓒ $4.$ Ⓓ $8.$

7. The expression $\frac{1}{3}\sqrt{45}$ is equivalent to

Ⓐ $\sqrt{5}.$ Ⓑ $3\sqrt{5}.$

Ⓒ $5.$ Ⓓ $15.$

8. What are all the values for x such that $x^2 < 3x + 18$?

Ⓐ $x < -3$ Ⓑ $-3 < x < 6$

Ⓒ $x > -3$ Ⓓ $x < 6$

9. If $f(x) = 2x^3 - 18x$, what are all the values of x at which $f(x) = 0$?

Ⓐ $0, 3$ Ⓑ $-3, 0, 3$

Ⓒ $-6, 0, 6$ Ⓓ $-3, 2, 3$

10. Which of the following is equal to $\frac{17.5(10^{-2})}{500(10^{-4})}$?

Ⓐ $0.035(10^{-2})$ Ⓑ $0.35(10^{-2})$

Ⓒ $0.0035(10^{2})$ Ⓓ $0.035(10^{2})$

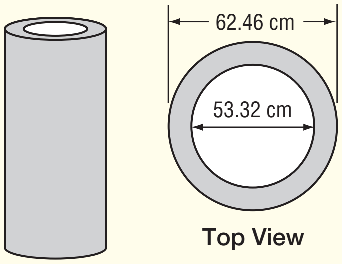
The Princeton Review — Test-Taking Tip

Question 7 You can use estimates to help you eliminate answer choices. For example, in Question 7, you can estimate that $\frac{1}{3}\sqrt{45}$ is less than $\frac{1}{3}\sqrt{49}$, which is $\frac{7}{3}$ or $2\frac{1}{3}$. Eliminate choices C and D.

Part 2 Short Response/Grid In

Record your answers on the answer sheet provided by your teacher or on a sheet of paper.

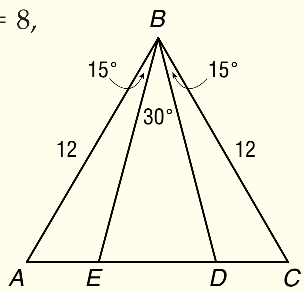
11. If the outer diameter of a cylindrical tank is 62.46 centimeters and the inner diameter is 53.32 centimeters, what is the thickness of the tank?



12. What number added to 80% of itself is equal to 45?

13. Of 200 families surveyed, 95% have at least one TV and 60% of those with TVs have more than 2 TVs. If 50 families have exactly 2 TVs, how many families have exactly 1 TV?

14. In the figure, if $ED = 8$, what is the measure of line segment AE?



15. If $a \leftrightarrow b$ is defined as $a - b + ab$, find the value of $4 \leftrightarrow 2$.

16. If $6(m + k) = 26 + 4(m + k)$, what is the value of $m + k$?

17. If $y = 1 - x^2$ and $-3 \leq x \leq 1$, what number is found by subtracting the *least* possible value of y from the *greatest* possible value of y?

18. If $f(x) = (x - \pi)(x - 3)(x - e)$, what is the difference between the greatest and least roots of $f(x)$? Round to the nearest hundredth.

Part 3 Quantitative Comparison

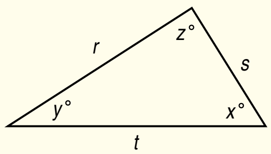
Compare the quantity in Column A and the quantity in Column B. Then determine whether:

- Ⓐ **the quantity in Column A is greater,**
- Ⓑ **the quantity in Column B is greater,**
- Ⓒ **the two quantities are equal, or**
- Ⓓ **the relationship cannot be determined from the information given.**

19. $-1 < xy < 0$

Column A	Column B
$x + y$	xy

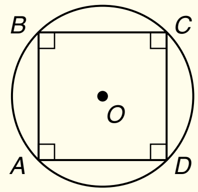
20.



$z = x + y$

Column A	Column B
t	$\sqrt{r^2 + s^2}$

21.



circumference of circle $O = 8\pi$

Column A	Column B
perimeter of square $ABCD$	16

22.

$$x - y + z = 5$$
$$x + y + z = 9$$

Column A	Column B
$x + z$	6

23. $nx \neq 0$

Column A	Column B
$-2nx$	$(x - n)^2$

www.algebra2.com/standardized_test

CONTENTS